On Configuration Spaces and Modules over Little Discs Operad

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Moscow 2016

Abstract

We describe combinatorial dg-Hopf coloured cooperadic models for configuration spaces of points in first quarter and in n-sided polygon. Using first model we obtain another proof of version of Kontsevich formality theorem for two subspaces in vector space and extend formality morphism to a G_∞ one.

1 Introduction

Our work is inspired by the papers [5] and [7]. In [7] Thomas Willwacher constructed combinatorial cooperadic model for Swiss-Cheese operad and proved that every L_∞ stable formality morphism $T_{poly} \to D_{poly}$ can be up to homotopy extended to G_∞. We extend his results for two different configuration spaces.

Firstly, we construct cooperadic model for the configuration space of points in the first quarter. This configuration space is important for stable formality morphism for vector space with two subspaces (branes). In [5] authors constructed a L_∞ stable formality morphism $U : T_{poly} \to CC^\bullet(Cat_\infty(A, B, K))$. We describe another L_∞ stable formality morphism, such that restriction to $CC^\bullet(A)$ we obtain the Kontsevich stable formality morphism. We prove that this morphism up to homotopy can be extended to G_∞ one.

Secondly, we construct cooperadic model for the configuration space of points in n-sided polygon. We believe that this model can be used for the study of configuration space of points on Riemann surfaces with punctures and in conformal field theory.

The structure of the paper is as follows. In section 2 we desribe configuration space of points in the first quarter and operadic structure on it. Section 3 describes combinatorial 4-coloured dg-Hopf cooperad A. In section 4 we construct a map (Kontsevich space integral I) of 4-coloured dg-Hopf cooperads between A and semi-algebraic forms on configuration spaces. Section 5 is devoted to the proof that I is quasi-isomorphism. In section 6 we recall key notions from A_∞ framework. In section 7 we construct L_∞ stable formality morphism $U : T_{poly} \to CC^\bullet(Cat_\infty(A, B, K))$ and prove that it can be extended to a G_∞ one. Section 8 describes how to adapt the constructions of sections 2–5 to the configuration space of points in polygon.

2 Configuration space

Definition 1. Let $C(n, m, k)$ be a configuration space of $n + m + k$ points in the first quarter of real plane without origin, such that m points belong to x-axis, k points belong to y-axis and n points don't belong to either of the axes. For convenience we consider an element of $C(n, m, k)$ as n points of type I, m points of type II, k points of type III and distinguish additional point 0 in the origin. On this configuration space there is the natural action of $\mathbb{R}^+$, $(\lambda(X))(p) = \lambda X(p)$, where $X \in C(n, m, k)$ is a configuration and p is a number of point from 1 to $n+m+k$. We define $QC(n, m, k)$ as factor $C(n, m, k)/\mathbb{R}^+$.

For every pair of numbers a, b from 1 to $n+m+k$ we construct a map $\omega_{a,b} : QC(n, m, k) \to S^1$, such that $\omega_{a,b}(X) = \frac{X(a)-X(b)}{|X(a)-X(b)|}$. In other words to every pair of points we assign the direction of vector between them.

For every ordered triple of numbers a, b, c we construct a map $\Theta_{a,b,c} : QC(n, m, k) \to [0; +\infty]$, such that $\Theta_{a,b,c}(X) = \frac{|X(a)-X(b)|}{|X(a)-X(c)|}$. With this maps we have the embedding $i : QC(n, m, k) \to (S^1)^N \times [0; +\infty]^M$ with N – number of pairs and M – number of ordered triples.

Definition 2. The closure of $i(QC(n,m,k))$ is denoted by $\overline{QC}(n,m,k)$ and is called Fulton-Macpherson compactification.

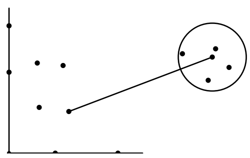
Proposition 1. The Fulton-Macpherson compactification $\overline{QC}(n,m,k)$ is homotopically equivalent to the $QC(n,m,k)$.

We define in the same way Fulton-Macpherson compactifications $C(n)$ of points in R^2 and $SC(n,m)$ with n points in the upper half-plane and m points on x-axis.

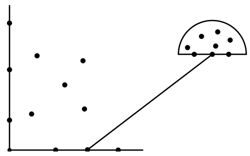
2.1 Decomposition of the boundary of the compactification

For the short we write $QC(V)$ instead $\overline{QC}(n,m,k)$. The element X in the boundary of compactification $QC(V)$ is determined by some clusters of points, such that for any points a,b in the cluster and c out we have $\Theta_{a,b,c}(X) = 0$. More specifically let Y be a configuration in $QC(n,m,k)$ and X_i be the configuration, obtained from Y by reducing i times all distances between points in this cluster. Then X is the limit of Y_i. We have 4 types of clusters:

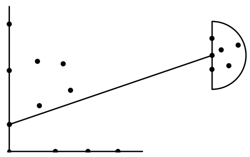
(i) The cluster match to the set of points, converged to the point of type I. Therefore all this points should be type one too. This strata has a form of $QC(n-l+1,m,k)\times C(l)$, when $C(l)$ is the configuration space of l points in $\mathbb{R}^2$ modulo translations and multiplication by scalar. On the figure the cluster of points converges to type I point.



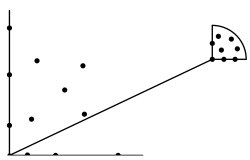
(ii) The cluster match to the set of points, converged to the point of type II. Therefore all this points should be types I or II. This strata has a form of $QC(n-l,m-t+1,k)\times SC(l,t)$, when $SC(l,t)$ is the configuration space of l points in $\mathbb{R}^2$ and t points on the x-axis modulo translations and multiplication by scalar.



(iii) The cluster match to the set of points, converged to the point of type III. Therefore all this points should be types I or III. This strata has a form of $QC(n-l,m,k-t+1)\times SC(l,t)$, when $SC(l,t)$ is the configuration space of l points in $\mathbb{R}^2$ and t points on the y-axis modulo translations and multiplication by scalar.

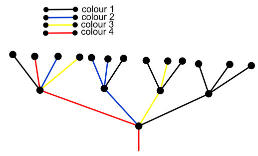


(iv) The cluster match to the set of points, converged to the 0. Points in this cluster may be any type. This strata has a form of $QC(n-l,m-t,k-s)\times QC(l,t,s)$.



Of course, we can have some different clusters and this clusters may be included one in another. By the set of clusters and configurations in each of them we can uniquely obtain our configuration.

Now we are going to define the specific coloured graphs responsible for the classification of boundary strata of $\overline{QC}(n,m,k)$. In the graph we have the unique root vertex *root*. We define the length $l(v)$ of a vertex v as the number of edges in the minimal path from this vertex to the root. Because considered graphs are trees every path is minimal and therefore each edge connects two vertices with length differed by one. We call the vertex v external if for any connected by an edge vertex w we have $l(w) < l(v)$. It follows that every external vertex has only 1 edge (in other case we have a cycle, considered of two edges from v to w and w' and paths from w and w' to the root vertex). We call internal all other vertices except for the root vertex. For every vertex v we call the unique edge e between v and w output if $l(w) < l(v)$. It is convenient to add the output edge *out* to the root vertex too. One can see that every path from the vertex to the root is a sequence of outputs and therefore every edge is output for one of the vertices. If e, e' are two consequent edges in such path we call e' an ancestor of e. Let *out* be ancestor to every other edge with the endpoint *root*. Now the admissible coloured tree is a tree with all output edges for every vertex coloured in one of 4 colours (we often say that the vertex is coloured in the same colour) satisfying the following conditions:



A typical element of admissible coloured graphs.

(i) The ancestor of the edge of colour 4 can be only the edge of colour 4.
(ii) The ancestor of the edge of colour 2 can be only the edge of colour 2 or 4.
(iii) The ancestor of the edge of colour 3 can be only the edge of colour 3 or 4.
(iv) The edge *out* is colour 4.
We call the admissible coloured tree (n,m,k)-tree if it has n external vertex of colour 1, m of colour 2 and k of colour 3.

Lemma 1. Boundary strata of $\overline{QC}(n,m,k)$ are in $1-1$ correspondence with (n,m,k) trees.

Proof. We associate to every boundary strata of $\overline{QC}(n,m,k)$ an admissible coloured tree in the following way. For every point we assign the external vertex coloured in the colour associated to the type of point, i.e. colour 1 to the points in the first quarter, 2 to the points on x-axis and 3 – on y-axis. For a configuration in the open part of $\overline{QC}(n,m,k)$ we assign graphs without internal vertices. For every cluster of type i we add an internal vertex v of colour i and connect vertices corresponded to the points in cluster to v. In such way for every set of clusters we obtain the unique graph. On the other hand by the same rules from every (n,m,k) tree we can construct the set of clusters and the corresponding boundary strata. Therefore this construction gives $1-1$ correspondence between (n,m,k)-graphs and boundary stratas of $\overline{QC}(n,m,k)$. □

Corollary 1. The collection of spaces $\{\overline{QC}, \overline{SC}, \overline{SC}, \overline{C}\}$ can be endowed with the structure of the 4-coloured operad.

Proof. We have 4 colours and therefore 4 different type of actions. The first colour operad action is the composition with $\overline{C}(l)$. For the second, third and fourth components of operad it is the usual composition in the operad of small discs (in 4-th case) or in the Swiss-Cheese operad. For the forst component we define $\circ_i^1 : \overline{QC}(n,m,k) \otimes \overline{C}(l) \to \overline{QC}(n+l-1,m,k)$ as replacing i-th type I point, by the configuration space $C(l)$. For the boundary strata we replace i-th external vertex in the associated graph by the subgraph of l external and 1 internal vertices of colour 1.

The second colour operad action is zero on the 3-d and 4-th components of the operad. On the second one it is usual operadic composition in the Swiss-Cheese operad. On the first component we have $\circ_i^2 : \overline{QC}(n,m,k) \otimes \overline{SC}(l,t) \to \overline{QC}(n+l, m+t-1, k)$ where $m \geqslant i$ is mapping two configurations to the boundary strata of $\overline{QC}(n+l, m+t-1, k)$ of type $\overline{QC}(n,m,k) \times \overline{SC}(l,t)$ replacing ith type II point, by the configuration space $SC(l,t)$. For the boundary strata we replace i-th external vertex in the associated graph by the subgraph of l external vertices of colour 1 and t external and 1 internal of colour 2.

The third colour operad action is zero on the 2-d and 4-th components of the operad. On the third one it is usual operadic composition in the Swiss-Cheese operad. On the first component we have $\circ_i^3 : \overline{QC}(n,m,k) \otimes \overline{SC}(l,t) \to \overline{QC}(n+l, m, k+t-1)$ where $k \geqslant i$ is mapping two configurations to the boundary strata of $\overline{QC}(n+l, m, k+t-1)$ of type $\overline{QC}(n,m,k) \times \overline{SC}(l,t)$ replacing ith type III point, by the configuration space $SC(l,t)$. For the boundary strata we replace i-th external vertex in the associated graph by the subgraph of l external vertices of colour 1 and t external and 1 internal of colour 3.

The fourth colour operad action is non-zero only on the first component of the operad: $\circ^4 : \overline{QC}(n,m,k) \otimes \overline{QC}(l,t,s) \to \overline{QC}(n+l, m+t, k+s)$ is mapping two configurations to the boundary strata of $\overline{QC}(n+l, m+t, k+s)$ of type $\overline{QC}(n,m,k) \times \overline{QC}(l,t,s)$, replacing 0 by the configuration space $\overline{QC}(l,t,s)$. For the boundary strata we replace i-th external vertex in the associated graph by the subgraph of appropriate number of external vertices and 1 internal of colour 4.

The associativity condition implies from the one for the coloured graphs. □

Proposition 2. The image of i consecutive applying of partial operad compositions is a submanifold with corners codimension i.

Proof. The image of i operadic compositions is included in the boundary strata corresponded to the graph with i internal vertices. Such graphs correspond to the strata form of $X_1 \times X_2 \times \ldots \times X_i$. It remains to prove that each product reduces the dimension by one. For all components without first it is well-known. For the first one we have:

The dimension of $\overline{QC}(n,m,k)$ is $2n+m+k-1$. The dimension of $C(l)$ is $2l-3$. The dimension of $SC(l,t)$ is $2l+t-2$. Let us consider the dimension of the image of each partial composition:

$$(i) dim(\overline{QC}(n,m,k) \times \overline{C}(l)) = 2n+m+k-1+2l-3 = 2(n+l-1)+m+k-2 = \\ = dim(\overline{QC}(n+l-1,m,k)) - 1.$$

$$(ii) dim(\overline{QC}(n,m,k) \times \overline{SC}(l,t)) = 2n+m+k-1+2l+t-2 = \\ = 2(n+l)+(m+t-1)+k-2 = dim(\overline{QC}(n+l,m+t-1,k))-1 = dim(\overline{QC}(n+l,m,k+t-1))-1.$$

$$(iii) dim(\overline{QC}(n,m,k) \times \overline{QC}(l,t,s)) = 2n+m+k-1+2l+t+s-1 = \\ = 2(n+l)+(m+t)+(k+s)-2 = dim(\overline{QC}(n+l,m+t,k+s)) - 1.$$

□

The collection of algebras $\{\Omega(\overline{QC}(n,m,k)),\Omega(\overline{SC}(n,m)),\Omega(\overline{SC}(n,k)),\Omega(\overline{C}(n))\}$ of semi-algebraic forms on the corresponded topological spaces has induced cooperadic structure and de-Rham differential. It is easy to see, that they are compactible with each other and give the structure of dg-Hopf 4-coloured cooperad. Now we can formulate the main problem of our paper.

Question 1. Find the algebraic-combinatorical model for

$$\Omega := \{\Omega(\overline{QC}(n,m,k)),\Omega(\overline{SC}(n,m)),\Omega(\overline{SC}(n,k)),\Omega(\overline{C}(n))\},$$

i.e. the dg-Hopf 4coloured cooperad A and map $I : A \to \Omega$ such that I is quasi-isomorphism compactible with product, differential and cooperad structure.

It will be useful to describe more suitable decomposition of the boundary in the faces codimension 0. Let A be a subset of V, i.e. a collection (l,s,t), such that $l \leqslant n$, $s \leqslant m$ and $t \leqslant k$. It is convenient to call points from A external and points from $V\backslash A$ - internal. We have a canonical projection $\pi : QC(V) \to QC(A)$ forgetting all points not from A.

Proposition 3. For every V the space $QC(V)$ is manifold with corners. Moreover, the projection map $\pi : QC(V) \to QC(A)$ is the semi-algebraic map.

Proof. The proof is parallel to the case of configuration space of points in R^2. For more details we refer to the [5, subsection 5.9]. □

Proposition 4. The boundary of $QC(V)$ is the union of images of operadic compositions $\circ_i^j : QC[U] \otimes QC[W] \to QC(V)$, such that $|W| \geqslant 2$ and $|U| \geqslant 1$. For the $j = 4$ we permit W to be cardinality 1. Here we embed $C(X)$ and $SC(X)$ in $QC(X)$ for the abuse of notation. This images are codimension 0 faces of $QC(V)$ and their intersection is at least codimension 1.

Proof. The proof of the statement is parallel to the one of the [5, Proposition 5.11]. □

2.2 The decomposition of the corresponding to π fiberwise boundary.

Let us describe the decomposition of the corresponding to π fiberwise boundary.

Recall that $QC^{\delta}(V) = \overline{(\delta QC(V)) \cap \pi^{-1}(QC(A)/\delta QC(A))}$.

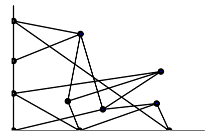
We call the operadic partial composition $\circ_i^j : QC[U] \otimes QC[W] \to QC(V)$, where $j < 4$ "good" if $A \subset W$ or $|A \cap W| \leqslant 1$ and all external points in W are of type j. In other words the subset W may contain only one external vertex if the composition is applied in this point. The partial composition $\circ^4$ is called "good" if $A \subset U$ or $A \subset W$, because this composition is always applied in the point 0.

Proposition 5. The images of "good" partial compositions give a stratification of $QC^{\delta}(V)$ into codimension 0 faces. More specifically, each image is codimension 0 strata, an intersection of every two of them is at least codimension 1 and every configuration belong to one of these stratum.

Proof. This statement is the adapted to our problem version of the [5, Proposition 5.20]. The only difference in the proof is that we can compose configuration $QC(W)$ with 1 external point by the 4-th colour action. □

3 Graphs

3.1 Graphs with external vertices



A typical element of $Gr(5,2,3)$.

Now we can start the definition of the sought-for collection of algebras A. Firstly, let us define the space of graphs $Gr(m,n,k)$. The graph of type (n,m,k) in our convention is the set of n numerated vertices in the first quarter (type I), m on the x-semiaxis (type II), k on the y-semiaxis (type III) and distinguish vertex 0 in the origin and ordered oriented edges between them. We prohibit loops, multiple edges, edges between points on the same axis (include 0) and edges, started in a not type I vertex and ended in a type I one. Let $Gra(m,n,k)$ be the set of all graphs of type (n,m,k) and $Gr(V)$ be the vector space over $\mathbb{R}$, generated by $Gra(m,n,k)$. From now on we write $V(\Gamma)$ and $E(\Gamma)$ for the sets of vertices and edges of the graph Γ. We have the natural action of $S := S_{E(\Gamma)} \times {\mathbb{Z}/2\mathbb{Z}}^{E(\Gamma)}$ on $G(m,n,k)$. $S_{E(\Gamma)}$ acts by permutations on the set $E(\Gamma)$ with the sign of the permutation. The action of ${\mathbb{Z}/2\mathbb{Z}}^{E(\Gamma)}$ is given by inversing of the edges with the sign. Let us consider the space of coinvariants $G(n,m,k) := Gr(n,m,k)^S$. In other words for every graph in $Gr(n,m,k)$ we can choose its orientation by numerating and orientating all edges and obtain an graph representing an element of $G(n,m,k)$.

Corollary 2. Any graph $\Gamma \in Gr(n,m,k)$ consisting a loop or multiple edges is equal to 0 in $G(n,m,k)$.

Proof. (i) If Γ contains a loop e then we have an action of $\mathbb{Z}/2\mathbb{Z}$ by inversing this edge e. So $\Gamma = -\Gamma \Rightarrow \Gamma = 0$.

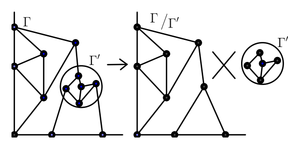
(ii) If Γ contains two edges e_1 and e_2 with the same ends then we have an action of S_2 by permutations of the numbers of these edges. So $\Gamma = -\Gamma \Rightarrow \Gamma = 0$. □

In the same way we define $Graphs_2(n)$ as graphs with m and k equal 0 and without vertex 0. According to the [5] the collection of $\{Graphs_2(n)\}$ is endowed with the structure of cooperad. By considering graphs without 0 and type III (respectively type II) vertices we obtain Gra_{SC} (respectively Gra'_{SC}). From [7] we know that $\{Gra_{SC}, Graphs_2\}$ (resp. $\{Gra'_{SC}, Graphs_2\}$) is endowed with the structure of 2-coloured operad.

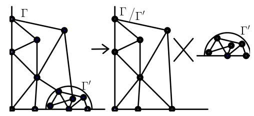
Proposition 6. The collection of vector spaces $\{G, Gra_{SC}, Gra'_{SC}, Graphs_2\}$ can be endowed with the structure of 4-coloured operad.

The cocomposition action is given by fragmentation of the graph into subgraph clusters. We demand second color action to act trivially on third and fourth component, third – on second and fourth and fourth – on all except fourth. The action on $Gra_{SC}, Gra'_{SC}, Graphs_2$ are as in [7]. We need to show the action of operadic cocomposition on the first component.

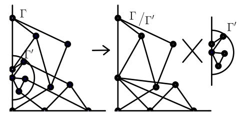
(i) Let Γ be a graph and Γ' be a subgraph of Γ consisted only of l type I vertices and edges between them. Then we have a map $\circ^1 : G(m,n,k) \to G(m-l+1,n,k) \otimes Graphs_2(l)$, such that $\circ^1(\Gamma) = \Gamma/\Gamma' \otimes \Gamma'$. By Γ/Γ' we understand graph with the set of vertices $V(\Gamma/)V(\Gamma')$ with type I vertex p corresponded to Γ' and with the set of edges $E(\Gamma)/E(\Gamma')$, where the edge with one end at $V(\Gamma')$ now has this end at the vertex p.



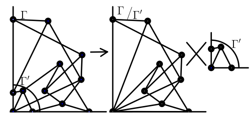
(ii) We have the map $\circ^2 : G(m,n,k) \to G(m-l, n-t+1, k) \otimes Gra_{SC}(l,t)$, such that $\circ^2(\Gamma) = \Gamma/_{\Gamma'} \otimes \Gamma'$, when Γ' consists of l vertices of type I and $t \geqslant 1$ vertices of type II. We demand p to be type II in this case. Of course any type III vertex can not be included in subgraph Γ', because type III vertex can not be infinitesimal to type II one.



(iii) We have the map $\circ^3 : G(m,n,k) \to G(m-l, n, k-s+1) \otimes Gra'_{SC}(l,0,s)$, such that $\circ^3(\Gamma) = \Gamma/_{\Gamma'} \otimes \Gamma'$, when Γ' consists of l vertices of type I and $s \geqslant 1$ vertices of type III. We demand p to be type III in this case. Analogously any type II vertex can not be included in the subgraph Γ, because type II vertex can not be infinitesimal to type III one.



(iv) We have the map $\circ^4 : G(m,n,k) \to G(m-l, n-t, k-s) \otimes G(l,t,s)$, such that $\circ^4(\Gamma) = \Gamma/_{\Gamma'} \otimes \Gamma'$, but now Γ' consists of l vertices of type I, t vertices of type II and s vertices of type III. We demand p to be 0 in this case. In all this cases we can obtain a non-admissible graph (for example with double edges). All such graphs we equate to 0.



3.2 Differential forms on configuration spaces

Let us consider configuration spaces with two points. We want to assign for every case one-dimensional form ω on the appropriate configuration space. In all cases we can embed this points in C and consider them as complex numbers x and y. Let $\omega^+(x,y) := \frac{1}{2\pi} d\left(arg(\frac{x-y}{\bar{x}-y})\right)$ and $\omega^-(x,y) = \omega^+(y,x)$. We put $\omega = \omega^+ + \omega^-$. There are 4 different cases:

(i) $QC(2,0,0) \simeq C(2) \simeq S^1$, and the map of the last homeomorphism is $(x,y) \to arg(y-x)$. Then

$$\omega = \frac{1}{2\pi} d(arg(y-x) - arg(\bar{y}-x) + arg(x-y) - arg(\bar{x}-y)) =$$
$$= \frac{1}{2\pi} d(arg(y-x) - arg(\bar{y}-x) + arg(y-x) + \pi + arg(y-\bar{x})) = \frac{1}{\pi} d(arg(y-x))$$

- double standard volume form on S^1. One can see that $\int_{QC(2,0,0)} \omega = 2$.

(ii) $QC(1,1,0)$. Let x be a point on the line. We have a function $\phi^+ = \frac{1}{2\pi} arg(\frac{y-x}{\bar{y}-x})$ measuring the hyperbolic angle between x and y in H. This function is invariant by translation and scalar multiplication and gives an isomorphism $QC(1,1,0) \simeq (0,1)$. One can see that $\omega^+ = d\phi^+$. On the other hand $\omega^- = \frac{1}{2\pi} arg(\frac{x-y}{x-y}) = 0$. From the statement above it is immediate then $\int_{QC(1,1,0)} \omega = 1$.

(iii) $QC(1,0,1)$. Let x be a point on the line. We have a function $\phi^- = \frac{1}{\pi} arg(y-x)$ measuring the angle between y and x. This function is invariant by translation and scalar multiplication and gives an isomorphism $QC(1,0,1) \simeq (0,1)$. By the reasons in (i) we have $\omega = d\phi$. From the statement above it is immediate then $\int_{QC(1,0,1)} \omega = 1$.

(iv) $QC(0,1,1)$. Let x be on x-axis and y on y-axis. We have a function $\phi = \frac{1}{2\pi} arg(\frac{y-x}{y+x})$. This function is invariant by scalar multiplication and gives an isomorphism

$QC(1,0,1) \simeq (0, \frac{1}{2})$. We have

$$\omega = \frac{1}{2\pi} d(arg(y-x) - arg(-y-x) + arg(x-y) - arg(x-y)) = \frac{1}{2\pi} d(arg(y-x) - arg(y+x) + \pi) = $$
$$= \frac{1}{2\pi} d(arg(\frac{y-x}{y+x}) + \pi) = d\phi$$

From the statement above it is immediate then $\int_{QC(1,0,1)} \omega = \frac{1}{2}$.

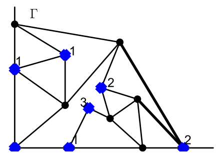
We assign to every graph $\Gamma \in G(m,n,k)$ the differential form $\omega_\Gamma \in \Omega(QC(n,m,k))$, such that $\omega_\Gamma = \wedge_{e\in E} \omega_e$, where ω_e is defined in this way. Let s and f be two ends of this edge. We can consider projection $\pi : QC(V) \to QC(s,f)$, where $QC(s,f)$ is configuration space of two points of the same types as s and f. Let $\omega_e := \pi * \omega$, where ω is defined as above.

Let $c_\Gamma := \int_{QC(n,m,k)} \omega_\Gamma$. It is important that $c_\Gamma \neq 0$ only if

$$2n + m + k - 1 = dim(QC(n,m,k)) = deg\omega_\Gamma = |E(\Gamma)|.$$

3.3 Graphs with external and internal vertices

Further we are interested in the twisted in the sense of [2] version of G. We have a natural action of $S_{l,t,s} := S_l \times S_t \times S_s$ on the vector space $G(n+l, m+t, k+s)$ as follows. S_t and S_s permutate the last t (respectively s) vertices of the type II (respectively III) with the sign of the permutation. S_l permutate the last l first type vertices with trivial sign. The twisted version of the operad $\{G(m,n,k)\}$ is the collection of vector spaces $\{TwG(n,m,k) := \sum_{l,t,s} G(n+l, m+t, k+s)^{S_{l,t,s}}\}$. In this graphs we have (n,m,k) numerated vertices called external and any amount of vertices without numeration called internal. As the matter of fact we take the sum of all permutations of the internal vertices. For our purposes it is convenient to use "augmented" version of twisting, i.e we prohibit internal vertices with valency 0. The collection $\{TwG, TwGra_{SC}, TwGra'_{SC}, TwGraphs_2\}$ has induced from $\{G, Gra_{SC}, Gra'_{SC}, Graphs_2\}$ 4-coloured cooperadic structure.



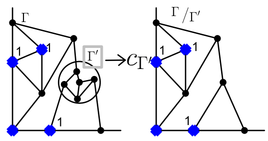
The example of a graph from $TwG(3,2,1)$ with $(4,1,1)$ internal vertices.

Proposition 7. For every (n,m,k) the vector space $TwG(n,m,k)$ has structure of commutative dg-algebra.

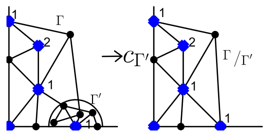
Proof. The multiplication is constructed as usual by gluing graphs on external vertices. Let $\Gamma \in TwG(n,m,k)$ be a graph with (l,t,s) internal vertices and k edges. Let introduce grading $deg(\Gamma) = k - 2l - t - s$. It is obvious that $deg(\Gamma) + deg(\Gamma') = deg(\Gamma \cdot \Gamma')$. Let us call the graph Γ "contractible" if Γ is connected and has at most one external vertex. Let $P(n,m,k) \subset TwG(n,m,k)$ be vector subspace generated by all "contractible" graphs. We have a map $c_\Gamma : P(n,m,k) \to \mathbb{R}$. Let us define the differential of Γ as follows. We have a full cocomposition map $\circ : TwG \to TwG \otimes TwG$. Let us consider the composition with the tensor product $TwG \xrightarrow{\circ} TwG \otimes TwG \xrightarrow{\otimes_P \mathbb{R}} TwG \otimes TwG \otimes_P \mathbb{R}$. We have an evaluation $e : TwG \otimes TwG \to TwG$ defined as $e(1 \otimes x) = e(x \otimes 1) = x$ and $e(x \otimes y) = 0$ with x and y from the augmentation ideal. We define d as the composition $d := TwG \xrightarrow{\circ} TwG \otimes TwG \xrightarrow{\otimes_P \mathbb{R}}$

$TwG \otimes TwG \otimes_P \mathbb{R} \xrightarrow{e \otimes_P \mathbb{R}} TwG \otimes P \xrightarrow{\otimes_P \mathbb{R}} TwG$. In this differential we have summands of images of the cooperadic action with one of the tensor factors to be from P. From this definition one can see that the differential has 5 pieces:

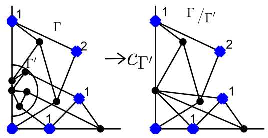
(i) If e is an edge and at least one of its ends is type I vertex, then we can contract this edge with the sign. $d_1(\Gamma) = 2\sum_{e \in Contr(\Gamma)}(-1)^e \Gamma/_e$, where $Contr$ is the set of all contractible edges and $(-1)^e$ is dependent on the parity of number of edge e. This piece of the differential decrease number of internal type I vertices and edges by 1, so has grading 1.



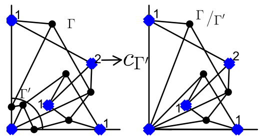
(ii) Let Γ' be a subgraph with all vertices types I and II, such that at most one of them is external (necessarily type II). Then we can contract this subgraph to type II vertex with coefficient. $d_2 = \sum_{\Gamma'} c_{\Gamma'} \Gamma/_{\Gamma'}$, where c will be defined before the part about twisted operad. The coefficient c is not 0 only if $deg(\Gamma') = -2$. Therefore this piece has grading 1.



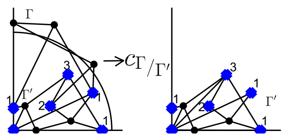
(iii) Let Γ' be a subgraph with all vertices types I and III, such that at most one of them is external (necessarily type III). Then we can contract this subgraph to type III vertex with coefficient. $d_2 = \sum_{\Gamma'} c_{\Gamma'} \Gamma/_{\Gamma'}$, where c will be defined before the part about twisted operad. The coefficient c is not 0 only if $deg(\Gamma') = -2$. Therefore this piece has grading 1.



(iv) Let Γ' be a subgraph consisting 0 and with some internal vertices. All external vertices in Γ' except 0 are prohibited. Then we can contract this subgraph to 0 with coefficient. $d_4 = \sum_{\Gamma'} c_{\Gamma'} \Gamma/_{\Gamma'}$. The coefficient c is not 0 only if $deg(\Gamma') = -1$. Therefore this piece has grading 1.



(v) Let Γ' be a subgraph consisting all external vertices. Then we can reduce all to this subgraph. $d_5 = \sum_{\Gamma'} -c_{\Gamma/_{\Gamma'}} \Gamma'$. The coefficient c is not 0 only if $deg(\Gamma/_{\Gamma'}) = -1$. Therefore this piece has grading 1. The whole differential $d = d_1 + d_2 + d_3 + d_4 + d_5$ has grading 1 because of all his pieces have grading 1.



□

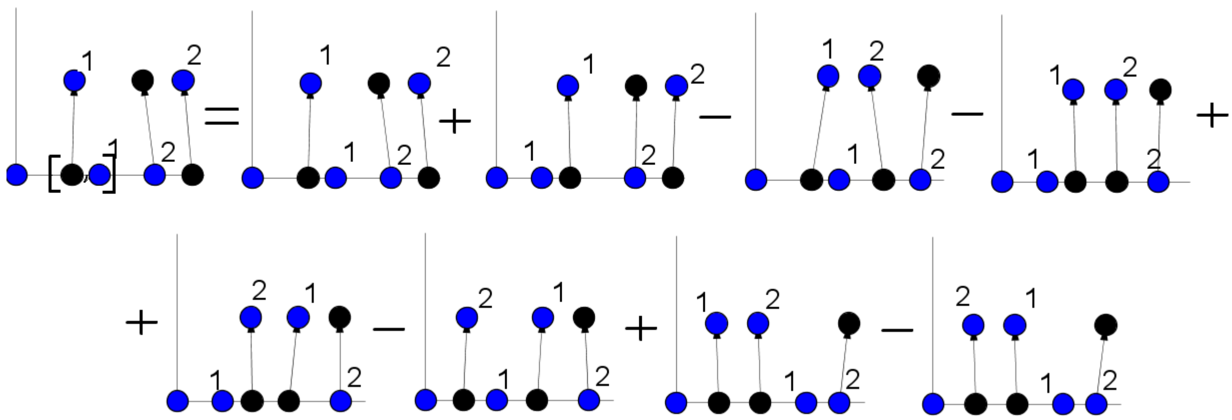
Remark 1. After applying differential to our graph Γ we can obtain a graph with multiple edges or edges between points on one axis. We equate such graphs to 0.

Corollary 3. $\{TwG, TwGra_{SC}, TwGra'_{SC}, TwGraphs_2\}$ is 4-coloured dg-Hopf cooperad.

Proof. The structure of dg-commutative algebra should be shown for every component of the cooperad. We have done it for the first component in the Proposition 7. For all other components there are well-known results, we refer to [5] and [7]. □

3.4 Lie decorated graphs

Let Gr_1 be the vector space generated by graphs with n external vertices and m edges to type II vertices. Our goal in this section is to change the basis of the Gr_1. More specifically we generate the vector space Gr_2 by graphs of new basis and prove that this two vector spaces are isomorphic. Let us introduce the notion of Lie decorated graph. Now we can consider Lie brackets on the points on one axis, such that in any Lie cluster we have at most one external vertex. For the Lie decorated graph we associate a sum of usual graphs as follows. Between two vertices of a cluster we cannot have any external vertices except consisting in this Lie cluster. Now we have two types of internal vertices, free (without cluster) and fixed (in cluster). For a fixed vertices we have the sum of two graphs (first with first vertex (or maybe bracket) before second one from this Lie bracket and second - after). For every case we take a sum of all graphs obtained in the same way. For a free one we have the sum of graphs, obtained by consecutive transpositions this vertex with other with the sign. It is easier to understand with an example.



Let us define Gr_2 as the vector space generated by all Lie decorated graphs with n external vertices and m edges to type II vertices.

Proposition 8. Vector spaces Gr_1 and Gr_2 are isomorphic. Therefore Lie decorated graphs are another basis of the space of graphs.

Proof. Let V be m-dimensional vector space, generated by edges $v_1, \dots, v_m$ to type II vertices. We can associate to the graph $\Gamma \in Gr_1$ with k internal vertices an element of the $T^{k+n}(S^\cdot V)_{polylin}$, where index *polylin* means that the product of all elements is $v_1 v_2 \dots v_m$, in the following way. Let i-th vertex on the line has edges $v_{i_1}, v_{i_2}, \dots, v_{i_l}$. We associate to this vertex an element $x_i := v_{i_1} v_{i_2} \dots v_{i_l} \in S^\cdot(V)$. If i-th vertex is zero-valent we associate to it an element $1 \in S^\cdot V$. To the graph we associate $x_1 \otimes x_2 \otimes \dots \otimes v_{k+n}$. To the element of $T^{k+n}(S^\cdot V)_{polylin}$ we can associate graph in the same way. Therefore we have $Gr_1 \simeq \oplus_{k \geqslant 0} T^{k+n}(S^\cdot V)_{polylin}$. Let us first consider the case $n = 0$, when all vertices are internal. We have the natural actions of S_k on Gr_1 by permutating vertices. Then we have $Gr_1 \simeq \oplus_{k \geqslant 0} T^k(S^\cdot V)_{polylin} \simeq \oplus_{k \geqslant 0} T^k(S^\cdot V)_{polylin} \otimes_{k[S_k]} k[S_k] = \oplus_{k \geqslant 0} T^k(S^\cdot V)_{polylin} \otimes_{k[S_k]} Assoc(k)$.

Lemma 2. Operad *Pois* governing Poison algebras is associated graded operad to operad *Assoc* governing associative algebras.

Proof. We have a canonical map $\pi : Assoc \to Com$, induced from the projections $k[S_n] \to k$ on the space of n-ary operations. Let I be the kernel of this map. We have a natural filtration $F_0 \supset F_1 \supset F_2$on $Assoc$, $F_0 := Assoc$, $F_1 := I$, $F_2 := I^2, \dots$. Let us consider the graded operad associated to this filtration. $gr(Assoc) := F_0 \big/ F_1 \oplus F_1 \big/ F_2 \oplus \dots$. Now we claim that the induced product $x \cdot y : F_i \big/ F_{i+1} \times F_j \big/ F_{j+1} \to F_{i+j} \big/ F_{i+j+1}$ becomes commutative. The ideal I is generated by the bracket $[x, y] := xy - yx$. And if $x \in F_i, y \in F_j$ we have $[x, y] \in F_{i+j+1}$. Therefore $x \cdot y - y \cdot x \in F_{i+j+1}$. On the other hand

$\{x,y\} = x\cdot y - y\cdot x \in F_{i+j+1}\Big/F_{i+j+2}$ defines Poisson bracket. One can check the anti-commutativity, Jacobi and Leibniz identities. So we have a map $Pois \to gr(Assoc)$. The associative product can be uniquely determined from induced commutative one and Poisson bracket, so this map is surjection. From equality of dimensions of $Assoc(n)$ and $Pois(n)$ follows that $gr(Assoc) = Pois$. $\square$

Therefore as vector spaces:

$$Gr_1 \simeq \oplus_{k\geqslant 0}T^k(S^{\cdot}V)_{polylin}\otimes_{k[S_k]} Assoc(k) \simeq \oplus_{k\geqslant 0}T^k(S^{\cdot}V)_{polylin}\otimes_{k[S_k]} Pois(k) \simeq Gr_2.$$

The last isomorphism is clear, because $\otimes_{k[S_k]}Pois(k)$ is the same as endowing graphs with poison bracket on them.

Now assume that graph has n external vertices of type II. Then we have a natural action of $S_k\times S_n$ by permutating internal and external vertices. Therefore

$$Gr_1 \simeq \oplus_{k\geqslant 0}T^{k+n}(S^{\cdot}V)_{polylin}\otimes_{k(S_k\times S_n)}k(S_{k+n}) = \oplus_{k\geqslant 0}T^{k+n}(S^{\cdot}V)_{polylin}\otimes_{k(S_k\times S_n)}Assoc(k+n)$$
$$\simeq \oplus_{k\geqslant 0}T^{k+n}(S^{\cdot}V)_{polylin}\otimes_{k(S_k\times S_n)}Pois(k+n) = \oplus_{k\geqslant 0}T^{k+n}(S^{\cdot}V)_{polylin}\otimes_{k(S_k)}Pois(k+n)^{S_n}.$$

We take coinvariants to the action of S_n and therefore the poisson bracket on external vertices is identically 0. Therefore right hand side corresponds to Poisson bracket on graphs such that any two external points are in different Lie clusters. This graphs generate Gr_2. $\square$

We can change basises in such way on both axes.

3.5 Admissible graphs

Definition 3. The graph is called externally disconnected if there is a connected subgraph consisting only of internal vertices such that all vertices of types II and III are free. We denote vector space, generated by all externally disconnected graphs $\Gamma \in G(n,m,k)$ as $N(n,m,k)$.

Proposition 9. For every externally disconnected graph Γ with at least one external vertex we have $c_\Gamma = 0$.

Proof. Let $\Gamma' \subset \Gamma$ be externally disconnected connected component. Then $\Gamma = \Gamma' \cup \Gamma$" and Γ" is not empty.

$$c_\Gamma = \int_{QC(V(\Gamma))}\omega_\Gamma = \int_{QC(V(\Gamma))\to QC(V(\Gamma'))}\int_{QC(V(\Gamma'))}\omega_{\Gamma'}\omega_{\Gamma"} =$$
$$= \int_{QC(V(\Gamma))\to QC(V(\Gamma'))}\omega_{\Gamma"}\int_{QC(V(\Gamma'))}\omega_{\Gamma'} = c_{\Gamma'}\int_{QC(V(\Gamma))\to QC(V(\Gamma'))}\omega_{\Gamma"}.$$

$$c_\Gamma = \int_{QC(V(\Gamma))}\omega_\Gamma = \int_{QC(V(\Gamma))\to QC(V(\Gamma"))}\int_{QC(V(\Gamma"))}\omega_{\Gamma'}\omega_{\Gamma"} =$$
$$= \int_{QC(V(\Gamma))\to QC(V(\Gamma"))}\omega_{\Gamma'}\int_{QC(V(\Gamma"))}\omega_{\Gamma"} = c_{\Gamma"}\int_{QC(V(\Gamma))\to QC(V(\Gamma"))}\omega_{\Gamma'}.$$

From the formula above coefficient c_Γ is not 0 only if both $\omega_{\Gamma'}$ is top form on the $QC(V(\Gamma'))$ and $\omega_{\Gamma"}$ is top form on $QC(V(\Gamma'))$. Therefore

$$deg(\omega_\Gamma) = deg(\omega_{\Gamma'}) + deg(\omega_{\Gamma"}) = \dim(QC(V(\Gamma'))) + \dim(QC(V(\Gamma"))) = \dim(QC(V(\Gamma))) - 1.$$

So ω_Γ is not top form and $c_\Gamma = 0$. $\square$

Proposition 10. The vector space $N(n,m,k)$ is an ideal in $G(n,m,k)$ closed under differential.

Proof. The multiplication of two graphs is constructed by gluing graphs. Therefore if $\Gamma' \subset \Gamma_1$ is externally disconnected connected component then $\Gamma' \subset \Gamma_1 \cdot \Gamma_2$ will be externally disconnected connected component too.

Let $\Gamma \in N(n,m,k)$ be externally disconnected graph.

When we contract an edge between two type I vertices the resulting graph is externally disconnected.

Every other summand in the differential $d\Gamma$ has form $c_{\Gamma_1}\Gamma_2$. It is immediate that Γ_1 or Γ_2 is externally disconnected. If Γ_2 is externally disconnected then this summand belong to $N(n,m,k)$. If Γ_1 is externally disconnected then it has external vertices because Γ has and $c_{\Gamma_1} = 0$ by Proposition 9. □

Let us recall the notion of non-admissible graphs on Gra_{SC} (and therefore Gra'_{SC}) and $Graphs_2$ from [7] and [5] respectively. One can see that this vector spaces are obtained from the restriction N on appropriate graphs. We denote this sets as N_{SC}, N'_{SC} and N_2 respectively.

Proposition 11. The collection $N_{all} := \{N, N_{SC}, N'_{SC}, N_2\}$ is cooperadic ideal in $Tw_{all} := \{TwG, TwGra_{SC}, TwGra'_{SC}, TwGraphs_2\}$, namely $\circ(N_{all}) \subset N_{all} \otimes Tw_{all} + Tw_{all} \otimes N_{all}$.

Proof. For all components without first one it is well known fact and we refer to [5] and [7]. It is enough to prove this fact for $N \subset G$. Let $\Gamma \in N(n,m,k)$ be externally disconnected graph and $\Gamma' \subset \Gamma$ be its externally disconnected connected component. Every cocomposition action sends Γ to the product $\Gamma_1 \times \Gamma_2$, such that intersection of Γ' with at least one of this graphs is not empty set. Then this intersection is externally disconnected connected component in this graph. □

Corollary 4. Factors $A_{QC} := TwG\big/_{N}$, $A_{SC} := TwGra_{SC}\big/_{N_{SC}}$, $A'_{SC} := TwGra'_{SC}\big/_{N'_{SC}}$ and $A_2 := TwGraphs_2\big/_{N_2}$ are well defined and induces from the Tw_{all} the structure of 4-coloured dg-Hopf cooperad on the collection $A := \{A_{QC}, A_{SC}, A'_{SC}, A_2\}$.

4 Kontsevich space integral

Our goal is to construct the map I from the $TwG(n,m,k)$ to the semi-algebraic forms on $\overline{QC}(n,m,k)$. Let $\Gamma \in G(n+l, m+t, k+s)$ be a graph represented an element of $TwG(n,m,k)$. Then we have a natural projection $\pi : \overline{QC}(n+l, m+t, k+s) \to \overline{QC}(n,m,k)$ forgetting all internal points. Let $I(\Gamma) := \pi_* \omega_\Gamma$. It is useful to rewrite $I(\Gamma)$ as the integral along the fiber. $I(\Gamma) := \int_{\overline{QC}(n+l,m+t,k+s) \to \overline{QC}(n,m,k)} \omega_\Gamma$.

Theorem 1. The map I commutes with the 4-coloured cooperad structures on $\{A\}$ and $\{\Omega\}$.

Proof. The proof is parallel to the [5, Section 9.5]. □

Proposition 12. I is the chain map.

Proof. Let $\Gamma \in G(n+l, m+t, k+s)$ be a graph represented an element of $D(n,m,k)$. $I(d\Gamma) = \sum_{i=1}^{5} I(d_i\Gamma)$. On the other side we have $dI(\Gamma) = d\int_{\overline{QC}(n+l,m+t,k+s) \to \overline{QC}(n,m,k)} \omega_\Gamma =$

$\int_{\overline{QC}^{\delta}(n+l,m+t,k+s)\to\overline{QC}(n,m,k)} \omega_\Gamma$ by the Stokes formula. We have from statement above a decomposition of $\overline{QC}^{\delta}(n+l,m+t,k+s)$. So $dI(\Gamma) = \sum_{\circ_i^j} \int_{QC(U)\times QC(W)\to\overline{QC}(n,m,k)} \omega_\Gamma$, where $\circ_i^j$ is "good" partial composition and U, W are defined as in statement above. Let us consider three cases:

(i) $A \subset W$

$$\int_{QC(U)\times QC(W)\to\overline{QC}(n,m,k)} \omega_\Gamma = \int_{QC(U)\times QC(W)\to QC(W)} \int_{QC(W)\to\overline{QC}(n,m,k)} \omega_\Gamma =$$
$$= c_{\Gamma/\Gamma'} \int_{QC(W)\to\overline{QC}(n,m,k)} \omega_{\Gamma'} = c_{\Gamma/\Gamma'} I(\Gamma')$$

by the Fubini theorem. This summand is a part of $I(d_5\Gamma)$.

(ii) $j < 4$ and $|A\cap W| \leqslant 1$

$$\int_{QC(U)\times QC(W)\to\overline{QC}(n,m,k)} \omega_\Gamma = \int_{QC(U)\times QC(W)\to QC(U)} \int_{QC(U)\to\overline{QC}(n,m,k)} \omega_\Gamma =$$
$$= c_{\Gamma'} \int_{QC(U)\to\overline{QC}(n,m,k)} \omega_{\Gamma/\Gamma'} = c_{\Gamma'} I(\Gamma/\Gamma')$$

by the Fubini theorem. This summand is a part of $I(d\Gamma)$with use only of $d_1+d_2+d_3$. This is obvious from the definition to the cases, when W consist at least 1 not type I vertex. In this case we refer to the [5, Section 9.4]

(iii) $j = 4$, $A\cap W = \varnothing$

$$\int_{QC(U)\times QC(W)\to\overline{QC}(n,m,k)} \omega_\Gamma = \int_{QC(U)\times QC(W)\to QC(U)} \int_{QC(U)\to\overline{QC}(n,m,k)} \omega_\Gamma =$$
$$= c_{\Gamma'} \int_{QC(U)\to\overline{QC}(n,m,k)} \omega_{\Gamma/\Gamma'} = c_{\Gamma'} I(\Gamma/\Gamma')$$

by the Fubini theorem. This summand is a part of $I(d_4\Gamma)$.

On the other hand, to all summands in $I(d\Gamma)$ we can assign such "good" operadic composition that $\int_{QC(U)\times QC(W)\to\overline{QC}(n,m,k)} \omega_\Gamma$ gives our summand. Therefore $I(d\Gamma) = dI(\Gamma)$ □

Proposition 13. The map I is the map of algebras, i.e. commute with multiplications.

Proof. The proof is described in the [5, Section 9.2]. □

Proposition 14. The map I vanishes on N.

Proof. We refer to [5] and [7] for all components except first one. Let $\Gamma \in N(n,m,k)$ be externally disconnected graph and $\Gamma' \subset \Gamma$ be its externally disconnected connected component. Then $\Gamma = \Gamma1\cdot\Gamma2$, where Γ_1 is obtained from Γ' by adding all external vertices without edges. We know that I is morphism of algebras. Therefore it is enough to prove that $I(\Gamma_1) = 0$ Let A be the set of external vertices, $V := V(\Gamma_1)$, $W := V(\Gamma') = (l,t,s)$. Then map $\pi : QC(V)\to QC(A)$ factors as $QC(V) \xrightarrow{p} QC(W)\times QC(A) \xrightarrow{q} QC(A)$ where q is the projection on the second factor. We know that dim $QC(W) = 2l+t+s-1$ and immediately dim $(q^{-1}(x)) = 2l+t+s-1$. On the other side dim $QC(V) = 2(n+l)+m+t+k+s-1$ and dim $QC(A) = 2n+m+k-1$. Therefore dim $(\pi^{-1}(x)) = 2l+t+s >$dim $(q^{-1}(x))$. Now we refer to the proposition 8.14 in "Real homotopy theory of semialgebraic sets". □

As a consequence of the statements above we have well defined map $I : A \to \Omega$, compatible with the 4-coloured dg-Hopf cooperad structure.

Theorem 2. The map $I : A \to \Omega$ is quasi-isomorphism.

Proof. It is enough to prove it for the dual operad. We know that this map is quasi-isomorphism for all components of operad except for $A(n, m, k)^*$. We devote next section to the computation of cohomology of this component of the operad. □

5 Cohomology of graphs

5.1 Associated spectral sequence

Let us consider the filtration on graphs from $A(n, m, k)$ by the number of internal type I vertices. We have the associated spectral sequence $E_{p,q}$. This spectral sequence is bounded above and on the left, so this spectral sequence converges to cohomology of $A(n, m, k)$. On the 0 sheet we have the differential not changed the quantity of internal type I vertices. Any piece of the differential does not contain external vertices, so type I vertices are not involved in d_0.

Lemma 3. The first sheet $E^1_{p,q}$ is generated by graphs Γ satisfying two following conditions:
(i) All internal vertices of types II and III are univalent.
(ii) All external vertices of types II and III have valency 0.

Proof. Our goal is to compute cohomology of $(A(n, m, k), d_0)$. For this purposes we consider another filtration by the number of internal type II vertices and the associated spectral sequence $F_{p,q}$. The differential d_{00} on $F^0_{p,q}$ is contracting two points of type III to one. Due to [7, Section 5] on the first sheet we have the conditions of the lemma for types III vertices. For the computation d_{01} on $F^1_{p,q}$ we consider the filtration by the number of all internal vertices and consider the differential contracting two points of type II. Because of the same argument the cohomology satisfies conditions of the lemma. For the completeness of the proof we want to sketch a proof of the used statement from [7].

Proposition 15. The cohomology of $A(n, m, k)$ with the differential contracting 2 vertices of type II, at least one of which is internal, to one is generated by graphs Γ satisfying two following conditions:
(i) All internal vertices of type II are univalent.
(ii) All external vertices of type II have valency 0.

Proof. We firstly give a proof for the case when all vertices of type II are internal.

Let us consider only graphs with m edges ending at points of type II. We fix an order on this edges and denote as V vector space generated by edges $v_1, \ldots, v_m$. We denote by S^+V the augmentated algebra, associated to $S^\bullet V$, i.e. the kernel of the augmentation map $S^\bullet V \to k$. For edges $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ incoming in a vertex we associate an element $v_{i_1} v_{i_2} \ldots v_{i_k} \in S^k V$. For a graph Γ with m points of type II we assign an element of $[S^+V \otimes S^+V \otimes \ldots \otimes S^+V]_{polylin}$, where supscript *polylin* means that the product of all elements is $v_1 v_2 \ldots v_m$. Let us look on the action of differential, contracting two consequent points. If we have element $x_1 \in S^+V$ for the first of them and $x_2 \in S^+V$ for the second, the resulting element is $x_1 x_2 \in S^+V$. So we have a differential $S^+V \otimes S^+V \otimes \ldots \otimes S^+V \to$

$S^+V \otimes S^+V \otimes \ldots \otimes S^+V$, acting as Hochschild differential and $S^+V \otimes S^+V \otimes \ldots \otimes S^+V$ is complex. It is immediate that $[S^+V \otimes S^+V \otimes \ldots \otimes S^+V]_{polylin}$ is subcomplex. Thus to find cohomology on first sheet of the associated spectral sequence is the same as to find homology of the complex, obtained from $[S^+V \otimes S^+V \otimes \ldots \otimes S^+V]_{polylin}$ by taking coinvariants of S_m. The functor of taking coinvariants of finite group commutes with homology functor. Therefore it is enough to find homology of $[S^+V \otimes S^+V \otimes \ldots \otimes S^+V]_{polylin}$. It is subcomplex in $C_k = S^+V \otimes S^+V \otimes \ldots \otimes S^+V$. We can consider syzygy degree on this complex, $deg(x_1 \otimes x_2 \otimes \ldots \otimes x_k) = \sum_{i=1}^k deg(x_i) - k$. The differential increase grading by 1. This cochain complex is called bar-resolution $B(S^\bullet V)$.

Proposition 16. Koszul criterion. Let A be Koszul dual algebra and $A^{!'}$ be its Koszul dual coalgebra. Then the natural embedding $i : A^{!'} \to BA$ is quasi-isomorphism and the image of i has syzygy degree 0.

Proof. We refer to [6, Theorem 3.4.6]. □

It is well known (see [6, Example 3.2.5]) that $S^\bullet V$ and $\Lambda^\bullet V$ are Koszul dual algebras. Then cohomology of bar-resolution are trivial except for $H^0(BS^\bullet V)$ that is equal to shifted $\Lambda^\bullet V$ in this part. Cohomology of our complex is trivial except for $H^0 \simeq \Lambda^\bullet V_{polylin} \simeq k$. Note that syzygy degree of $x_1 \otimes x_2 \otimes \ldots \otimes x_n$ is equal to 0 only if $deg(x_i) = 1$. Therefore every vertex of type II is univalent. Cohomology elements are represented by the sum over all permutations of graphs with n internal univalent vertices.

Now we are ready to proof the proposition in general case. Let us consider vector space generated by all graphs with n external vertices and m edges ending at points of type II. It is a subcomplex with respect to d_0. Let us consider a filtration by the sum of valences of external vertices and the associated spectral sequence. It is bounded from left and above, so this spectral sequence converges to cohomology of this complex.

On the 0-th sheet of the associated spectral sequence we have a differential contracting two internal vertices. From the fact proved above we have that on the 1-st sheet all internal vertices are univalent.

On the 1-st sheet we have a differential, contracting one external and univalent internal vertices to external one. We want to prove that cohomology of the corresponding complex is represented by graphs with all external vertices of valence 0. Let us endow this complex with a grading, induced by filtration and denote the vector space, generated by graphs with the sum of valence of all external vertices k by A_k. We see that the differential reduces grading by 1. Our goal is to show that the cohomology of this complex is trivial in all parts except for H^0. Let us consider graphs Γ_i, obtained by Γ by replacing i-th external vertex by a Lie bracket with one internal univalent vertex. Then $d\Gamma_i = d\Gamma + X_i$, where X_i is the sum of graphs with the same vertices as Γ, but with the valency of one of external vertices is increased by one. One can note that X_i are linearly independent and therefore all graphs obtained from Γ by increasing valency of one external vertex by one are linear combination of X_i. Let us consider a sum of graphs $G = \sum G_i$ representing an element of $H^k, k > 0$ and let $Y = \sum Y_i$ be a sum of graphs, obtained from Y_i by reducing valency of one of external vertices by one. We can do the described above operation for every Y_i and obtain graphs $Y_{i,j}$. Then for $Z_{i,j} = \sum_{t \neq i} Y_i + Y_{i,j}$ we have $dZ_{i,j} = dY + X_{i,j}$. Since all $X_{i,j}$ are linearly independent we have $G = \sum Z_{i,j}$. Therefore $H^{>0} = 0$ and on the 2-nd sheet of the associated spectral sequence all ecternal vertices have valency 0. □

□

Let us look at the $E^1_{p,q}$. Here we have the differential, reduced the number of internal type I vertices by 1. d_1 consists of contracting of all edges, such that one of the ends is internal type I vertex and the piece of the fifth part of the differential with ${}^{\Gamma}/_{\Gamma'}$ consists from 0 and one internal type I vertex and of piece of the differential that put down internal vertex of the type I on the line (i.e. makes this vertex type II or III).

5.2 String complexes

Definition 4. Let s and t be two vertices. By Str_0 we denote the graph consisting of s, t and the edge e between them. Str_k is obtained from Str_{k-1} by replacing first edge (let it be from s to t') by internal type I vertex k and edges from s to k and from k to t'. Is the second edge is prohibited for Str_1 then we inverse it. The collection $\{Str_k\}$ for all k is called string from s to t.

Let d_1 be the piece of the differential d on $A(n,m,k)$, reducing the number of internal type I vertices by 1. We have three pieces of the differential on string complex:

(i) Contracting the edge between two vertices;

(ii) $c_{\Gamma/_{\Gamma'}}\Gamma'$, where Γ is obtained from Γ' by adding one vertex and an edge from it;

(iii) If at least one of the ends of string is of type I then we can put down this vertex on the line.

Proposition 17. $d_1^2(Str_k) = 0$.

Proof. $d_1^2(Str_k)$ decomposes as the sum of applying different pieces of the differential. We can think about two parts of piece (ii) as contracting 0-th and $k+2$-th edges. Then if we apply only (i) and (ii) pieces then the sum is zero, because we contract i-th edge after j-th and j-th after i-th with different signs. If we can apply piece (iii) twice, than the same argument gives 0. It is enough to consider the case when one applying of the differential put s on the line (the same for f). For contracting the edge not to s we have that this two pieces of the differential commute and cancel each other. All other summands are:

(i) Contracting $k+1$-th edge with coefficient 2, after that putting vertex f down;

(ii) Contracting $k+2$-th edge with coefficient 1, after that putting vertex f down;

(iii) Putting vertex f down and contracting $k+1$-th edge with the coefficient 1.

By our sign conventions the sum of all pieces is 0.

□

Definition 5. The complex generated by $\{Str_k\}$ and all differentials with the differential d_1 is called string complex.

Our goal is to compute the cohomology of string complexes for different types of s and t. (i) Points s and t belong to different axes.

Proposition 18. $d_1(Str^i_{2k+1}) = 2Str^i_{2k}$ and $d_1(Str^i_{2k}) = 0$ for $k \geqslant 0$.

Proof. Each edge between type I vertices is contracted with the coefficient 2. We have two pieces with coefficient 1 corresponding to the first edge and to the last edge (2 and 3 pieces of d). The piece (iii) of the differential does not act in this case.

$d(Str^i_k) = -Str^i_{k-1} + 2Str^i_{k-1} - \ldots + (-1)^k 2Str^i_{k-1} + Str^i_{k-1} = 2(1+(-1)^k)Str^i_{k-1}$. □

Therefore this string complex is acyclic.

(ii) Both points belong to one axis.

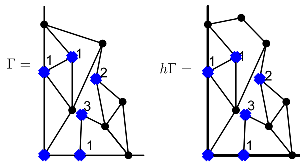
The only difference to the (i) is that $Str^{ii}_0 = 0$ and therefore cohomology is one-dimensional and spanned by Str^{ii}_1.

5.3 On the first sheet of the associated spectral sequence

Lemma 4. Let X be a sum of graphs Γ consisting at least one edge between type I and type II vertices, such that $d_1 X = 0$, then $X = dY + X'$ for some $Y \in A(n,m,k)$ and X' represents an element from the cohomology of the string complex between two vertices on the axes.

Proof. Let us introduce the operator $h : A(n,m,k) \to A(n,m,k)$.

If Γ does not consist any edge between type I and type II vertices then $h\Gamma := 0$. Let p be the leftmost II type vertex connected by edge to a type I one. From all such edges we can take the leftmost e. If the source of e is not internal bivalent type I vertex then let $h\Gamma$ be a graph, obtained from Γ by dividing this edge into 2 parts by adding internal type I vertex on it. If the source of e is internal bivalent type I vertex then we repeat the operation for this vertex. Let us consider the filtration F of the length of the string started from the considered leftmost vertex.



Remark 2. Elements in $A(n,m,k)$ are coinvariants with respect to the action of symmetric group by permutations of internal vertices. So the graph in $A(n,m,k)$ is the algebraic sum of the graphs differed by this permutation. For each of them the notion of leftmost is correctly defined. So it is correctly defined for the graph in $A(n,m,k)$.

Proposition 19. Let $\Gamma \in F^k(A)$. Then $d_1(h\Gamma) + hd_1\Gamma = \pm\Gamma + dY + \Gamma'$ with $\Gamma' \in F^{k+1}(A)$.

Proof. Let Γ be the graph with the length of the string of bivalent internal vertices k. Let us denote this string as $Str(k)$ and the remaining graph with the last point of the string Γ'. Let v be the last vertex of $Str(k)$.

(i) If v is vertex on one of the axes then we obtain an element Str_k^i or Str_k^{ii} from string complex. We drop upperscript for this case. $hStr_k = Str_{k+1}$ and $dStr_k = 0$.

From the subsection about string complexes we have $d_1(hStr_k) + hd_1Str_k = 2Str_k = Str_k + d(\frac{1}{2}Str_{k+1})$ except for $k = 1$ and two points are on one axis. In this case we have the cohomology element Str_1^{ii}.

(ii) If v is type I vertex.

Recall the clasification of the pieces of the differential from the subsection about string complexes. We know that valency of v is more than one, therefore pieces (ii) and (iii) does not act on the string. We have $d_1(\Gamma) = d(Str(k))\Gamma' + d(\Gamma)Str(k) = (-Str_{k-1} + 2Str_{k-1} - \ldots + (-1)^{k-1}Str_{k-1})\Gamma' + X = \pm Str_{k-1}\Gamma' + X$ with $X \in F^k(A)$.

$hd_1\Gamma = \pm hStr(k-1)\Gamma' + hX = \pm Str(k)\Gamma' + hX$.

$d_1(h\Gamma) = d_1(Str(k+1)\cdot\Gamma') = \mp Str(k)\Gamma' + Str(k+1)d\Gamma'$.

Then $d_1(h\Gamma) + hd_1\Gamma = hX + Str(k+1)d\Gamma' \in F^{k+1}(A)$. □

One can see that the number of edges is not changed with this operations. So from some m we have $F^m(M)$ is empty. But elements X, X', …, $X^{(m)}$ are homological. Therefore X is the sum of a appropriate string and dY.

□

The same argument can be applied in the proof of the analogous statement.

Lemma 5. Let X be a sum of graphs Γ consisting at least one edge between type I and type III vertices and 0, such that $d_1X = 0$, $X = dY + X'$ for some $Y \in A(n,m,k)$ and X' an element from a string complex between two vertices on the axes.

5.4 The ∞ sheet of the associated spectral sequence

The vector space C generated by all graphs with at least one edge between type I and not type I vertices is closed with respect to the differential d_1. The vector space DC generated by all graphs without any edges between type I and not type I vertices is closed with respect to the differential d_1 too. Therefore $E^1_{p,q} = C \oplus DC$ as complex. And two lemmas above claim that the contribution of C to the cohomology is trivial except cohomology of string complexes between points on the axes. And in all such graphs we have a component of points of only type I with all internal vertices valency at least 3. Due to [5] the cohomology of this space with differential d_1 is equal to $H^*(C(n))$.

Therefore all elements of $E^3_{p,q}$ are represented as graphs Γ such that all external vertices types II and III have valency 0, all internal vertices type II and III are univalent and paired by strings of length one with points of the same type and $d_1\Gamma = 0$, where d_1 acts on type I vertices by contracting an edge. The differential d_2 reduces the number of internal type I vertices by 2. One can see that d_2 contracts remaining strings with the coefficient $c_{Str_1} = 1$. Therefore this spectral sequence converges on the third sheet and the resulting graphs contain internal points only of type I and external points types II, III and 0 only valency 0. So the cohomology of graphs can be identified with $m! \otimes k! \otimes H^*(C(n)) \simeq H^*(C_1(m)) \otimes H^*(C_1(k)) \otimes H^*(C(n)) \simeq H^*(\overline{QC}(n,m,k))$ and map I is quasi-isomorphism.

6 A_∞ framework

6.1 A_∞ categories

Let us recall key moments from [1, Sections 2,3,4]

Definition 6. A small finite A_∞ category is a set of data:
(i) the finite set I (objects);
(ii) the element of $GrVect_k^{I\times I}$ A, i.e. $A = \{A_{a,b}\}, (a,b) \in I \times I$ (morphisms);
(iii) The codifferential d_A on $T_I(A[1])$.

Definition 7. A_∞ category with the set of objects of one element is called A_∞ algebra.

Proposition 20. Let A, B be two A_∞ algebras and $K - A_\infty - A - B-$bimodule. Let us consider $C_{a,a} = A$, $C_{a,b} = K$, $C_{b,a} = 0$ and $C_{b,b} = B$. Then C may be endowed with the structure of A_∞ category.

Proof. We need to define a codifferential d_A on $T_{\{a,b\}}(C[1])$. This codifferential is uniquely determined by its Taylor components $d^n_a : A^{\otimes n} \to A$, $d^n_b : B^{\otimes m} \to B$ and $d^{n,m}_{a,b} : A^{\otimes n} \otimes K \otimes B^{\otimes m} \to K$. We take $d_a = d_A$ and $d_b = d_B$ − codifferentials on A and B respectively. For $d^{n,m}_k$ we choose the morphism $d^{n,m,}_K : A^{\otimes n} \otimes K \otimes B^{\otimes m} \to K$, defined bimodule structure on K. For defined in such way d_C one can check that $d^2_C = 0$. □

Following [1] we denote this category as $Cat_\infty(A, B, K)$.

6.2 The Hochshild cochain complex for A_∞ category

Definition 8. Let us consider the space of $I \times I$ coderivations of the tensor algebra $CC(A) \coloneqq Coder_{I\times I}(T_I(A[1])) = Hom_{I\times I}(T_I(A[1], A[1])$. This space with natural grading is called Hochshild cochain complex of the A_∞ category A.

On the space of coderivations we have natural Lie bracket $[\bullet,\bullet]$. For our case we call it Gerstenhaber bracket. One can see that $CC(A)$ is actually cochain complex with $CC^n(A) = C^n(A,A) = Hom_{I\times I}(A[1]^{\otimes n}, A[1])$ and differential d as in usual Hochshild complex. With respect to bracket $[\bullet,\bullet]$ we may choose the Maurer-Cartan element γ, i.e. $[\gamma,\gamma]=0$ and differential $d_\gamma = [\gamma,\bullet]$ and obtain the structure of dg-Lie algebra on $CC(A)$. The usual Hochshild differential is obtained with γ - A_∞ structure in A.

Let us consider the case of $C = Cat_\infty(A,B,K)$. Then Hochshild cochain complex has a form $CC^n(C) = \oplus_{p+q=n} Hom^q(A^{\otimes p}, A) \oplus \oplus_{p+q+r=n} Hom^r(A^{\otimes p}\otimes K \otimes B^{\otimes q}, K) \oplus \oplus_{p+q=n} Hom^q(B^{\otimes p}, B)$. Let us denote the second summand as $C^n(A,B,K)$. Let us choose the Maurer-Cartan element $\gamma = d_A + d_K + d_B$ and an element $\phi = \phi_A + \phi_K + \phi_B$ of $CC^n(C)$. Then the differential $d_\gamma\phi = [d_A,\phi_A] + d_K\{\phi_A\} + [\gamma,\phi_K] + d_K\{\phi_B\} + [d_B,\phi_B]$, where $P\{Q_1,Q_2,\dots,Q_n\}$ is usual brace operation on Hochshild complex. One can see that $C^\bullet(A,B,K)$ is subcomplex in $CC(Cat_\infty(A,B,K))$.

We have two natural projections $p_A : CC(Cat_\infty(A,B,K)) \to CC(A)$ and $p_B : CC(Cat_\infty(A,B,K)) \to CC(B)$. It is immediate that both of them are chain maps.

Proposition 21. Maps p_A and p_B are L_∞ maps.

Proof. We refer to [1]. □

7 Formality morphisms

7.1 The framework of the research

Let us consider $X = R^d$ and two subspaces $U,V\subset X$, such that $X = U\cap V \oplus U\cap V^\perp \oplus U^\perp \cap V \oplus U^\perp\cap V^\perp$ and choose the basis $x_1,x_2,\dots,x_d$ compatible with this decomposition. To every pair of U and V we associate three graded vector spaces, namely

$$\begin{aligned}
A &= \Gamma(U,\wedge(\mathrm{N}U)) = \mathrm{S}(U^*)\otimes\wedge(X/U) = \mathrm{S}(U^*)\otimes\wedge(U^\perp\cap V)\otimes\wedge(U+V)^\perp,\\
B &= \Gamma(V,\wedge(\mathrm{N}V)) = \mathrm{S}(V^*)\otimes\wedge(X/V) = \mathrm{S}(V^*)\otimes\wedge(U\cap V^\perp)\otimes\wedge(U+V)^\perp,\\
K &= \Gamma(U\cap V,\wedge(\mathrm{T}X/(\mathrm{T}U+\mathrm{T}V))) = \mathrm{S}((U\cap V)^*)\otimes\wedge(U+V)^\perp,
\end{aligned}$$

Proposition 22. The set of graded vector spaces A,B,K may be endowed with the codifferential and generates an A_∞ category $Cat_\infty(A,B,K)$.

Proof. We endow A and B with the trivial A_∞ structure (every associative algebra is an A_∞ algebra in natural way). For the K we define $A_\infty - A - B$–bimodule structure by its Taylor components $d_K^{n,m}$ use graphs $Gr(0,n,m)$.

Let $\Gamma\in Gr(0,n,m)$ be a graph and $I : E(\Gamma)\to[1,\dots,d]$ be a labeling of the edges. We assign to Γ a morphism $O_\Gamma^K : A^{\otimes n}\otimes K\otimes B^{\otimes m}\to K$. Let us choose the element $a_1|a_2|\dots|a_n|k|b_1|\dots|b_m$. It is convenient to renumerate vertices, $1,\dots,n$ - vertices of II tpe with respect to the order on them, $n+1$ - vertex 0 and $n+2,\dots,n+m+1$ - vertices of type III with respect to their order. Let $\phi_i, i\in\{1,\dots,n+m+1\}$ be an element, corresponded to i-th vertex, i.e. a_i for $i\leqslant n$, k for $n+1$ and b_i for $i>n+1$. Let e be the edge in Γ. For every edge e with source i and target j we change ϕ_i by $<\phi_i, dx_{I(e)}>$ and ϕ_j by $\delta_{I(e)}\phi_i$. After that we multiplicate all ϕ_i as polyvector fields and project on K. We call the obtained map $O_I : A^{\otimes n}\otimes K\otimes B^{\otimes m}\to K$ and set $O_\Gamma^K(a_1|a_2|\dots|a_n|k|b_1|\dots|b_m) = c_\Gamma\sum_I O_I(a_1|a_2|\dots|a_n|k|b_1|\dots|b_m)$. By simmetrization we can consider the map O_Γ for $\Gamma\in G(0,n,m)$.

Proposition 23. The Taylor components $d_K^{n,m}(a_1|\dots|a_n|k|b_1|\dots|b_m) = \sum_{\Gamma\in G(0,n,m)} O_\Gamma$ define $A_\infty - A - B-$bimodule structure on K, coincides up to coefficient with the one in [1].

Proof. Tf we prove the coincidence up to coefficient of $d_K^{n,m}$ in our construction and in the one of [1] the proof of the first part of statement will follow from [1, Proposition 6.5]. Let $J := \{1, 2, \dots, d\}$ and J_1, $J_2 \subset I$ subsets, corresponding to the basis of U and V respectively. For every $L \subset J$ we write $L^\perp$ for $J - L$ and τ^L for the sum of operations, described for every edge e in the previous section for all $I : E(\Gamma) \to L$, i.e.

$$\tau_e^L(\phi_1 \otimes \phi_2 \otimes \dots \otimes \phi_i \otimes \dots \otimes \phi_j \otimes \dots \otimes \phi_{n+m+1}) = \\ = \sum_{k\in L} \phi_1 \otimes \phi_2 \otimes \dots \otimes < \phi_i, dx_k > \otimes \dots \otimes \delta_{x_k}\phi_j \otimes \dots \otimes \phi_{n+m+1}.$$

One can see that $O_\Gamma^K = c_\Gamma \prod_{e\in E(\Gamma)} \tau_e^J$. In the study of $d_K^{n,m}$ we are interested only in edges between vertices on two different axes.

Lemma 6. If s is vertex of type II and t of type III and e an edge from s to t then $\tau_e^J = \tau_e^{J_2\cap J_1^\perp}$ and $\omega_e = \frac{1}{2}\pi * \omega^{-,+}$, where $\omega^{+,-} = \frac{1}{2\pi} d(arg(\frac{(u-v)(u-\bar{v})}{(u+\bar{v})(u+v)}))$ is propagator on $QC(0,1,1)$ and π is projection on this space.

Proof. For the first part let us note that $< \phi_i, dx_k > \neq 0$ only if $x \in J_1^\perp$, because we can contract only vector fields with 1-forms. Therefore in $S(U*) \otimes \Lambda(U^\perp)$ we are interested only in skew-symmetric part.

We can differentiate along δ_{x_k} only symmetric part, so we are interested only in $S(V*)$ for point f. Therefore this operator gives not 0 only for $k \in J_2 \cap J_1^\perp$ and $\tau_e^J = \tau_e^{J_2\cap J_1^\perp}$.

To prove second part we should rewrite mentioned propagator for our case.

$$\omega^{-,+}(u, v) = \frac{1}{2\pi} d(arg(\frac{(u - v)(u - \bar{v})}{(u + \bar{v})(u + v)})) = \frac{1}{\pi} d(arg(\frac{(u - v)}{(u + v)} = 2\omega_e$$

□

Lemma 7. If s is vertex of type III and t of type II and e an edge from s to t then $\tau_e^J = \tau_e^{J_1\cap J_2^\perp}$ and $\omega_e = \frac{1}{2}\pi * \omega^{+,-}$, where $\omega^{+,-} = \frac{1}{2\pi} d(arg(\frac{(u-v)(u+\bar{v})}{(u-\bar{v})(u+v)}))$ is propagator on $QC(0,1,1)$ and π is projection on this space.

Proof. The proof is parallel to the one of lemma 7. □

Let us denote corresponding to [1] operators and differential forms as ω'_e, τ'_e and O'_Γ.

$$O_\Gamma^K = c_\Gamma \prod_{e\in E(\Gamma)} \tau_e^J = \int_{\bar{QC}(n,m,k} \prod_{e\in E(\Gamma)} \omega_e \tau_e^J = \frac{1}{2^{|E(\Gamma)|}} \int_{\bar{QC}(n,m,k} \prod_{e\in E(\Gamma)} \omega'_e \tau'_e = \frac{1}{2^{|E(\Gamma)|}} O'_\Gamma.$$

□

□

Now we can state the main result of this work.

Theorem 3. There exists a L_∞ quasi-isomorphism $U : T_{poly} \to CC^\bullet(Cat_\infty(A, B, K))$ and this quasi-isomorphism U up to homotopy can be extended to G_∞ quasi-isomorphism.

7.2 L_∞ quasi-isomorphism

The main goal of this section is to give explicit construction of the morphism U and prove that U is L_∞ quasi-isomorphism. We want to construct three L_∞ quasi-isomorphisms $U_A : T_{poly} \to CC^\bullet(A)$, $U_B : T_{poly} \to CC^\bullet(B)$ and $U_K : T_{poly} \to C^\bullet(A,B,K)$. Then $U = U_A + U_B + U_K$ is a L_∞ quasi-isomorphism between T_{poly} and $CC^\bullet(Cat_\infty(A,B,K))$.

Let us define U_A in the following way. Let $H(n,m)$ be the space generated by graphs with n type I vertices and m type II vertices with the same relations as in the definition of $Gr(n,m,k)$. Let $\Gamma \in H(n,m)$ be a graph and $I : E(\Gamma) \to \{1,\dots,d\}$ - labeling of the edges. We assign to this pair (Γ, I) a map $O_I : T_{poly}^{\otimes n} \to Hom(A^{\otimes m}, A)$ in the following way. For element $\gamma_1 \otimes \dots \otimes \gamma_n \in T_{poly}^{\otimes n}$ we assign to i-th type I vertex the element γ_i. The element O_I is defined in the same way as in the definition of $d_K^{n,m}$ with projecting on A. We set $O_\Gamma^A(a_1|a_2|\dots|a_n|k|b_1|\dots|b_m) = c_\Gamma \sum_I O_I(a_1|a_2|\dots|a_n|k|b_1|\dots|b_m)$. The map U_A is defined by its Taylor components $U_A^n : T_{poly}^{\otimes n} \to CC^\bullet(A)$, where $U_A^n = \sum_{\Gamma \in H(n,\bullet)} O_\Gamma^A$.

Theorem 4. The map $U_A : T_{poly} \to CC^\bullet(A)$ is L_∞ quasi-isomorphism.

Proof. Let dim $U = k$. One can see that A is algebra of functions $O(R^{k,d-k})$ of supermanifold $R^{k,d-k}$ and U_A is an Kontsevich formality morphism for this supermanifold. Thus according to super version of [4, Section 6.4] U_A is L_∞ quasi-isomorphism. □

We can repeat this procedure for graphs with vertices of type III instead II and subspace V instead U and obtain L_∞ quasi-isomorphism $U_B : T_{poly} \to CC^\bullet(B)$.

For the definition of U_K we consider graph $\Gamma \in G(n,m,k)$ and assign to it $O_\Gamma^K : T_{poly}^{\otimes n} \to Hom(A^{\otimes m} \otimes K \otimes B^{\otimes k}, K)$ in the same way as in the definition of $d_K^{m,k}$ with polyvector fields assigned to type I vertex. The Taylor component $U_K^n = \sum_{\Gamma \in G(n,\bullet,\bullet)}^{max} O_\Gamma^K$. In particular $d_K^{m,k}$ is O-th Taylor component U_K^0.

Theorem 5. The map $U = U_A + U_B + U_K : T_{poly} \to CC^\bullet(Cat_\infty(A,B,K)$ is a L_∞ morphism.

Proof. The proof of this statement is parallel to the one of [1, Theorem 7.2]. Recall from [1, Theorem 7.2] (iv) types of strata codimension 1 of $\bar{QC}(n,m,k)$. We discuss differences for each type of strata, appeared cause of considering different types of graphs.

(i) We have a cluster of points of type I converged to type I point. By Kontsevich lemma we may consider only graphs with two vertices. In our admissible graphs we have only one graph with non-trivial contribution, consisting of two points and one edge between them. The result will be only τ_e^J, corresponding to the Schouten–Nijenhuis bracket between poly-vector fields.

(ii) We have a cluster of points converged to a point of type II. This contribution for graphs is the same as contribution of $\sum_{I \sqcup I' = [n]} \pm U_K^{|I|}\gamma_I \bullet U_A^{|I'|}\gamma_{I'}$ for Hochschild complex.

(iii) We have a cluster of points converged to a point of type III. This contribution for graphs is the same as contribution of $\sum_{I \sqcup I' = [n]} \pm U_K^{|I|}\gamma_I \bullet U_B^{|I'|}\gamma_{I'}$ for Hochschild complex.

(iv) We have a cluster of points converged to a point 0. This contribution for graphs is the same as contribution of $\sum_{I \sqcup I' = [n]} \pm U_K^{|I|}\gamma_I \bullet U_K^{|I'|}\gamma_{I'}$ for Hochschild complex.

Therefore

$$\forall p,q \sum_{\Gamma \in (G(n,p,q))} \sum_{\delta_i \bar{QC}(n,p,q)} \int_{\delta_i \bar{QC}(n,p,q)} \omega_\Gamma = 0 \Rightarrow$$

$$\sum_{I \sqcup I' = [n]} \pm (U_K^{|I|}\gamma_I \bullet U_A^{|I'|}\gamma_{I'} + U_K^{|I|}\gamma_I \bullet U_B^{|I'|}\gamma_{I'} + U_K^{|I|}\gamma_I \bullet U_K^{|I'|}\gamma_{I'}) =$$

$$= \sum_{k \neq l} \pm U_K^{n-1}(\gamma_k \bullet \gamma_l, \gamma_1, \dots, \widehat{\gamma_i}, \dots, \widehat{\gamma_j}, \dots, \gamma_n)$$

Note that the right hand side equality is the condition for U to be L_∞. The left hand side formula can be rewritten by the Stokes formula.

$$\sum_{\Gamma\in(G(n,p,q))}\sum_{\delta_i \bar{QC}(n,p,q)}\int_{\delta_i \bar{QC}(n,p,q)}\omega_\Gamma = \sum_{\Gamma\in(G(n,p,q))}\int_{\bar{QC}(n,p,q)} d\omega_\Gamma = 0$$

For the last equality one should notice that ω_Γ is a product of closed forms and therefore is closed. Therefore U satisfies condition for L_∞ morphism. □

Theorem 6. The map U is a L_∞ quasi-isomorphism.

Proof. We have a commutative diagram of L_∞ morphisms: Theorem 5 claims that U_A and U_B are quasi-isomorphisms, so it is enough to prove that p_A or p_B is a quasi-isomorphism.

Lemma 8. The L_∞ morphism p_A is a quasi-isomorphism.

Proof. Theorem 3 claims that it is enough to show that the left derived action L_A is a quasi-isomorphism. The proof of this statement is identical to the one of [1, Proposition 7.5]. □

□

7.3 G_∞ morphism

Recall from [7] the definition of connected stable morphism.

Definition 9. A stable formality morphism U is called connected if the numbers c_Γ in its definition vanish on externally disconnected graphs.

It is easy to see the following statement.

Proposition 24. L_∞ morphism U, constructed in the previous section is connected.

Now we are ready to formulate the main result of this section.

Theorem 7. Let U be a connected L_∞ stable formality morphism. Then we can choose a homotopic stable formality morphism U' that can be extended to a G_∞ morphism.

To prove this fact we introduce some new notations.

Recall from [7] that $EGer$ is two-coloured operad governing two G_∞ algebras and G_∞ map between them.

Let $EELie$ be the coloured operad, governing L_∞ algebra X, A_∞ category of special type $Cat_\infty(A, B, K)$ and L_∞ morphism $X \to C^\bullet(Cat_\infty(A, B, K))$.

Let $EEGer$ be the coloured operad, obtained from $EELie$ by interchanging L_∞ algebra and morphism to G_∞.

Let us define $QCGraphs$ as graphs with all vertices of type I. We have a commutative diagram of operadic torsors:

Our stable formality morphism U after twisting gives a map $EELie \to QCGraphs$. We can choose corresponded to this formality morphism $QCGraphs^U$. Let us call the resulting coloured operad $HugeGraphs = (Br\ \ QCGraphs^U\ \ Graphs_2*)$.

Now we are ready for the proof of the theorem 8.

Proof. To finish the proof we should check that we can choose the homotopic one G_∞ morphism U' such that U' is stable formality morphism. It is parallel to the same fact in [7]. □

Corollary 5. Constructed in the subsection 7.2 L_∞ morphism U up to homotopy can be extended to the G_∞ one.

Proof. T_{poly} is Gerstenhaber algebra and therefore G_∞ algebra. On the Hochshild cochain complex of A_∞ category we have natural structure of Br algebra and therefore Br_∞ or equivalently G_∞ algebra. Thus the statement of the corollary is correct. Proposition 23 claims that U is connected stable formality morphism. Therefore this corollary is immediate consequence of theorem 8. $\square$

8 Algebraic model for points in n-sided polygon

Our goal in this section is to extend the algebraic model of graphs, described in section 3 to the case of configuration space of points in n-sided polygon. Proofs of all statements are paralel to the case described above, so this section is constructed as collection of statements and comments to them.

8.1 Configuration space of points in n-sided polygon

Definition 10. Let P be n-sided polygon with sides $p_1, \dots, p_n$. We denote by $PC_n(a_0, a_1, \dots, a_n)$ the configuration space of a_0 points in the interior of P and a_i points on p_i and by $\overline{PC_n}(a_0, a_1, \dots, a_n)$ its Fulton-Macpherson compactification. Points on a side p_i are called type i and points in the interior are called type 0.

Lemma 9. Boundary stratas of $\overline{PC_n}(a_0, a_1, \dots, a_n)$ are in $1-1$ correspondence with specified class of coloured trees.

Proof. Firstly, we need to define this class of trees. As in lemma 1 we associate each colour of edge to a type of codimension 1 strata, i.e. to a cluster of converging points. We have 3 cases:

(i) The cluster of points of type 0 converges to a point of type 0. Corresponding tree without internal vertices is coloured 0.

(ii) The cluster of points of types 0 and i converges to a point of type i. Corresponding tree is coloured a_i, we have n different colours.

(iii) The cluster of points of types 0, i and $i+1$ converges to vertex i. Corresponding tree is coloured b_i, we have n different colours. From here we understand index $n+1$ as 1 and a_0 (b_0) as a_n (b_n).

It follows that an ancestor of a vertex of colour a_i can be vertex of colour a_i, b_i, b_{i-1} or pre-root. An ancestor of a vertex of colour b_i can be only of colour b_i or pre-root. The colouration of pre-root vertex is fixed.

Now $1-1$ correspondence follows from arguments of lemma 1. $\square$

Corollary 6. The collection of spaces

$$\{\overline{PC}_n, \overline{QC}_1, \overline{QC}_2, \dots, \overline{QC}_n, \overline{SC}_1, \overline{SC}_2, \dots, \overline{SC}_n, \overline{C}\}$$

can be endowed with the $2n$+1-coloured operadic structure, where $\bar{C}_i$ is Fulton-Macpherson compactification of configuration space of points in the interior, $\overline{SC}_i$ - Fulton-Macpherson compactification of configuration space of points on i-th side and in the interior of polygon (quasi-isomorphic to Swiss-Cheese operad) and $\overline{SC}_i$ is defined as in definition 2 for vertex i and sides p_{i-1} and p_i.

Proof. On all components $\overline{SC}_i$ we have natural (Swiss-Cheese) operadic structures in colours a_i and 0. Let all other colours act by 0. On all components $\overline{QC}_i$ we have defined by corollary 1 operadic structures in colours b_i, a_i, a_{i-1} and 0. All other colours act by 0. On $\overline{PC}$ operadic action is defined by clusters parallel to Corollary 1. Associativity follows from the same reason. $\square$

Corollary 7. The collection of algebras

$$\Omega := \{\Omega(\overline{PC}_n), \Omega(\overline{QC}_1), \ldots, \Omega(\overline{QC}_n), \Omega(\overline{SC}_1), \ldots, \Omega(\overline{SC}_n), \Omega(\overline{C})\}$$

is endowed with the structure of $2n+1$-coloured dg-Hopf cooperad.

Note that propositions 2-5 can be adapted for our case without any changes.

8.2 Graph model

We consider graphs with external vertices of different types:

(i) type 0 - vertices in the interior of the polygon P;

(ii) type i - vertices on the side p_i;

(iii) vertex i - on the vertex i of the polygon.

We prohibit edges between vertices on one side. Consider the vector space generated by all graphs with a_0 vertices of type 0 and a_i of type i and take coinvariants as in section 3.1. We denote the resulting vector space $PG(a_0, a_1, \ldots, a_n)$.

Proposition 25. The collection of spaces

$$\{PG, G^1, G^2, \ldots, G^n, Graphs^1_{SC}, Graphs^2_{SC}, \ldots, Graphs^n_{SC}, Graphs_2\}$$

is endowed with the structure of $2n+1$-coloured cooperad.

Proof. The cooperadic structure on $Graphs^i_{SC}$ and $Graphs_2$ is the same as in Swiss-Cheese operad. All other colours we assume acting by 0. The cooperadic structure on G^i is defined in proposition 6. So we need to define a operadic composition on PG.

(i) For the action of the colour 0 (i.e. $PG \to PG \otimes Graphs_2$) we have the same map as in (i) in proposition 6;

(ii) For the action of a colour a_i (i.e. $PG \to PG \otimes Graphs^i_{SC}$) we have the same map as in (ii) in proposition 6;

(iii) For the action of a colour b_i (i.e. $PG \to PG \otimes G^i$) we have the same map as in (iii) in proposition 6;

$\square$

We consider augmented twisted version of this cooperad. Identically to proposition 7 we have the structure of commutative algebra on each component. Our next goal is to define differential forms on the configuration space of points in the polygon. We have

Proposition 26. There exists a map $f : H \to P$ called Schwarz–Christoffel mapping, such that f is conformal equivalence on the interior of P. If we extend to $H \cup R$, the image is P.

Let $a, b \in H$ and $\phi(a,b) = \frac{a-b}{\bar{a}-b}$ be the function measuring hyperbolic angle between two points in H. We have natural 1-form on configuration space of two points a and b in P. (Of course we should consider all cases o types of a and b, but the method is unique for all cases.) We can fix a by translation. $\omega_P(a,b) := d(\phi(f^{-1}(a), f^{-1}(b)))$.

Now for an edge e with endpoints s and t in a graph $\Gamma \in PG(a_0, a_1, \dots, a_n)$ we have 1-form $\omega_e := \pi * \omega_P(s,f)$, where π is projection of $PC(a_0, a_1, \dots, a_n)$ to appropriate configuration space with respect to types of s and f. Let $\omega_\Gamma = \prod_{e \in E(\Gamma)} \omega_e$ and $c_\Gamma = \int_{PC(a_0, a_1, \dots, a_n)} \omega_\Gamma$.

Let us define differential on $TwPG$ as in proposition 7. P is the vector space of contractible graphs, $c : P \to R$, $c(\Gamma) = c_\Gamma$ gives R a structure of P-module.

$$d := TwPG \xrightarrow{\circ} TwPG \otimes TwPG \xrightarrow{\otimes_P \mathbb{R}} TwPG \otimes TwPG \otimes_P \mathbb{R} \xrightarrow{e \otimes_P \mathbb{R}} TwPG \otimes P \xrightarrow{\otimes_P \mathbb{R}} TwPG.$$

It can be described explicitly as in proposition 7. It has three types of components:

(i) A contraction of an edge between two vertices of type 0;

(ii) A contraction of subgraph consisting only of types 0 and i vertices to vertex of type i;

(iii) A contraction of subgraph consisting only of types 0, $i-1$ and i vertices and vertex i to vertex i.

Combining all in this section we have

Proposition 27. The collection of spaces

$$\{TwPG, TwG^1, TwG^2, \dots, TwG^n, TwGraphs^1_{SC}, \dots, TwGraphs^n_{SC}, TwGraphs_2\}$$

is endowed with the structure of $2n+1$-coloured dg-Hopf cooperad.

We can define the space of externally disconnected graphs N_P as in definition 3 by assuming vertices on all sides to be free. One can check that propositions 9,10 and 11 are adapted for this case without any changes.

Corollary 8. The collection of vector spaces

$$PA := \{TwPG \big/ N_P, A^1_{QC}, A^1_{QC}, \dots, A^n_{QC}, A^1_{SC}, A^2_{SC}, \dots, A^n_{SC}, A_2\}$$

is well-defined and has induced structure of $2n+1$-coloured dg-Hopf cooperad.

Theorem 8. The Kontsevich space integral $I : PA \to \Omega$

(i) is correctly defined, i.e. vanishing on externally disconnected graphs;

(ii) commutes with the structures of coloured dg-Hopf cooperad;

(iii) is chain map;

(iv) is quasi-isomorphism.

Proof. The proof of each part this statement is identical to the one for configuration space of points in the first quarter. So we refer to theorems 1 and 2 and propositions 12,13 and 14. □

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