Unipotent representations from a geometric point of view (joint with Ivan Losev and Lucas Mason-Brown)

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Plan of the talk

1. Unipotent representations
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   - Vogan's desiderata for unipotent representations
2. Quantizations
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   - Canonical quantizations
3. Unipotent Harish-Chandra bimodules
4. Structure of unipotent Harish-Chandra bimodules
   - Generalized BVLS duality
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Unitary representations

G is a simple complex group.

Unitary representation is a pair $(\mathcal{H}, \rho)$, where
$\mathcal{H}$ – Hilbert space,
$\rho : G \to U(\mathcal{H})$ – continuous group homomorphism.

Question: [Gelfand, 1930-s]

Describe the set $\widehat{G}$ of irreducible unitary representations of G.

Solved for GL_n by Vogan in 1986, and for all other complex classical groups by Barbasch in 1989.

Orbit method

Let $\mathfrak{g}$ be the Lie algebra of G.
Idea (Kirillov, Kostant):

Expectation (orbit method):

There is a connection between the set of coadjoint orbits in $\mathfrak{g}^*$ and $\widehat{G}$.

On the left hand side of correspondence we have symplectic manifolds.
On the right hand side we have Hilbert spaces.

Hope:

The conjectured correspondence of the orbit method is given by "quantizing" the orbit.

The process of a "geometric quantization" producing a unitary representation out of a symplectic variety with symmetry is rather complicated. However, we can use a simpler notion of an algebraic quantization that will be defined later.

Unipotent representations

Let $\mathcal{O} \subset \mathfrak{g}^*$ be a nilpotent coadjoint orbit.
Note that the Killing form gives an identification $\mathfrak{g} \simeq \mathfrak{g}^*$, and in classical types for $G \subset GL_n$ nilpotent elements correspond to nilpotent matrices in $\mathfrak{g} \subset \mathfrak{gl}_n$

Hope/Expectation:

There is a finite set $Unip(\mathcal{O}) \subset \widehat{G}$ of irreducible unitary representation known as unipotent representations, associated with $\mathcal{O}$, satisfying certain good properties (to be discussed below).

Harish-Chandra bimodules

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

Definition:

A Harish-Chandra bimodule X is a $U(\mathfrak{g})$-bimodule, such that the adjoint action of $\mathfrak{g}$ integrates to the action of group G. We write $\mathrm{HC}(G)$ for the category of Harish-Chandra bimodules.

A Harish-Chandra bimodule is the same that a Harish-Chandra $(\mathfrak{g} \times \mathfrak{g}, G)$ Harish-Chandra module.
We want to have a notion of unitarizable Harish-Chandra bimodule V.
Recall that the real form produces an anti-holomorphic involution σ on $G \times G$, such that the diagonal copy $G \subset G$ is identified with $(G \times G)^{\sigma}$. We abuse the notation to denote the corresponding involution of $\mathfrak{g} \times \mathfrak{g}$ by σ.

Unitarizable Harish-Chandra bimodules

A Hermitian form on V is a sesquilinear pairing $\langle \bullet, \bullet \rangle : V \otimes V \to \mathbb{C}$, such that $\langle v, w \rangle = \overline{\langle w, v \rangle}$.
We say that a Hermitian form is σ-invariant if $\langle XvY, w \rangle = \langle x, -\sigma(Y) w \sigma(X) \rangle$ for any $X, Y \in \mathfrak{g}$.

Definition:

A HC-bimodule $V \in \mathrm{HC}(G)$ is unitarizable if V can be endowed with a non-degenerate σ-invariant Hermitian form $\langle \bullet, \bullet \rangle$ such that $\langle v, v \rangle > 0$ for any non-zero $v \in V$.

Deep result (Harish-Chandra):

The categories of unitary representations of G and of unitarizable HC-bimodules over G are equivalent.

Support of a Harish-Chandra bimodule

Consider $X \in \mathrm{HC}(G)$;
Set $J = \mathrm{Ann}(X) \subset U(\mathfrak{g})$ to be the annihilator of X;
$U(\mathfrak{g})$ has a PBW filtration, $F_i U(\mathfrak{g})$ is spanned by monomials of degree $\leqslant i$;
$\mathrm{gr}\, J \subset S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$;
For any ideal $I \subset \mathbb{C}[\mathfrak{g}^*]$ we can consider the associated variety $V(I) \subset \mathfrak{g}^*$ of points x, such that $f(x) = 0$ for any $f \in I$.

Fact: (Joseph)

$V(\mathrm{gr}\, J) = \overline{\mathcal{O}}$, where $\mathcal{O} \subset \mathfrak{g}^*$ is a nilpotent orbit.

Barbasch-Vogan-Luzstig-Spaltenstein duality

In 1985 Barbasch and Vogan constructed some interesting unipotent HC-bimodules. To define them we need a Barbasch-Vogan-Luzstig-Spaltenstein duality.

G – a simple complex Lie group;

$\mathfrak{g}$ – the Lie algebra of G;

$G^\vee$, $\mathfrak{g}^\vee$ – Langlands dual Lie group and Lie algebra.

$\mathcal{N} \subset \mathfrak{g}$, $\mathcal{N}^\vee \subset \mathfrak{g}^\vee$ – corresponding nilpotent cones.

BVLS duality:

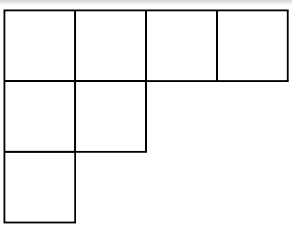
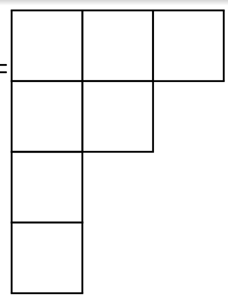
There is an order reversing map $d : \mathcal{N}^\vee/G^\vee \to \mathcal{N}/G$ on the sets of nilpotent orbits.
Orbits in the image of d are called *special* orbits, and d gives a bijection between the sets of special orbits.

BVLS duality

The description of BVLS duality is known for all simple Lie algebras $\mathfrak{g}$. Moreover, in classical types we have a combinatorial description using the parametrization of orbits by partitions.

Set $\mathfrak{g} = \mathfrak{sl}_n$. Nilpotent orbits in $\mathfrak{sl}_n$ are parametrized by partitions of n using the Jordan normal form. The BVLS duality corresponds to taking the transpose of a partition.

$$d(\alpha) = \alpha^T$$

$\alpha =$ , $d(\alpha) =$ 

All nilpotent orbits in $\mathfrak{sl}_n$ are special. That is not true for other types. For example, the minimal orbit in $\mathfrak{g}$ is always not special if $\mathfrak{g}$ is not of type A.

Special unipotent representations

$\mathcal{O} \subset \mathfrak{g}$ – a special orbit;

$\mathcal{O}^\vee \subset \mathfrak{g}^\vee$ is an orbit, such that $d(\mathcal{O}^\vee) = \mathcal{O}$;

$e^\vee, f^\vee, h^\vee$ – $\mathfrak{sl}_2$-triple for $\mathcal{O}^\vee$, $h^\vee \subset \mathfrak{h}^\vee$ is dominant.

$Z(U(\mathfrak{g})) \simeq S(\mathfrak{h})^W \simeq \mathbb{C}[\mathfrak{h}^*/W]$;

Recall that the maximal ideals in $U(\mathfrak{g})$ are parametrized by central characters, i.e. points in $\mathfrak{h}^*/W$;

Set $I(\mathcal{O}^\vee) = I(\frac{1}{2}h^\vee) \subset U(\mathfrak{g})$ to be the maximal ideal with central character $\frac{1}{2}h^\vee$.

Definition (Arthur, Barbasch-Vogan)

$Unip^s(\mathcal{O}) = \{X \text{ irred., } \mathrm{LAnn}(X) = \mathrm{RAnn}(X) = I(\mathcal{O}^\vee), d(\mathcal{O}^\vee) = \mathcal{O}\}$.

Limitations of the definition

1) If $\mathcal{O}$ is not special, then $Unip^s(\mathcal{O}) = 0$.
2) Set $\mathfrak{g} = \mathfrak{sl}_2$, and $\mathcal{O}$ to be the regular orbit in $\mathfrak{sl}_2$. We have $\frac{1}{2}h^\vee = 0$. The unique special unipotent representation is $\mathrm{Ind}_T^G \mathbb{C}$.
   However, we have two unitary representations of SL_2 with the central character $\frac{1}{2}$. Namely, set V^{even}, $V^{odd} \subset D(\mathbb{A}^1)$ to be the subspaces generated by monomials of even and odd degree respectively. Define the left and right actions of $\mathfrak{sl}_2$ on $D(\mathbb{A}^1)$ by:
   $$E = \frac{i}{2}x^2 \qquad H = x\frac{d}{dx} + \frac{1}{2} \qquad F = \frac{i}{2}\frac{d^2}{dx^2}.$$
   Note that V^{even} and V^{odd} are irreducible $\mathfrak{sl}_2$-bimodules under these actions. The adjoint action can be lifted to an action of the group SL_2, and both V^{even} and V^{odd} are unitary.
3) The example of 2) can be generalized to a unitary metaplectic representation of Sp_{2n}, associated with the minimal orbit in $\mathfrak{sp}_{2n}$. Such orbit is not special.

Vogan's desiderata

In 1987 Vogan proposed a list of desired properties of unipotent representations.

1) Each unipotent representation is a unitary representation associated to a nilpotent orbit $\mathcal{O}$.
2) For any unipotent representation X, $\mathrm{LAnn}_{U(\mathfrak{g})}(X) = \mathrm{RAnn}_{U(\mathfrak{g})}(X)$ is a maximal ideal.
3) All special unipotent representations are unipotent.
4) Unipotent representations satisfy Vogan's conjecture to be stated in the next slide.

Vogan's conjecture

Consider $X \in Unip(\mathcal{O})$;
For a good filtration on X the associated graded $\operatorname{gr} X$ is a finitely generated $S(\mathfrak{g})$-module;
$\operatorname{Supp}(\operatorname{gr} X) = \overline{\mathcal{O}}$.

Vogan's conjecture/theorem:

There is a good filtration on X and a homogeneous vector bundle M on $\mathcal{O}$, such that $\operatorname{gr} X \simeq \Gamma(\mathcal{O}, M)$ as representations of G.

The homogeneous vector bundle M can be roughly understood as the restriction of a $\mathbb{C}[\mathfrak{g}^*]$-module $\operatorname{gr} X$ to $\mathcal{O}$.

Quantizations of conical Poisson algebras

A – finitely generated Poisson algebra, i.e. commutative algebra with a Lie bracket satisfying Leibniz identity.

A admits an algebra grading $A = \bigoplus_{i=0}^{\infty} A_i$, $A_0 = \mathbb{C}$.

$\{A_i, A_j\} \subset A_{i+j-d}$ for a fixed integer $d > 0$.

Definition:

(Filtered) quantization of an algebra A is a pair $(\mathcal{A}, \theta)$, where

- $\mathcal{A} = \bigcup_i F_i\mathcal{A}$ is a filtered algebra;
- $[F_i\mathcal{A}, F_j\mathcal{A}] \subset F_{i+j-d}\mathcal{A}$;
- $\theta : \operatorname{gr}\mathcal{A} \to A$ – an isomorphism of graded Poisson brackets, where $\{a + F_{i-1}\mathcal{A}, b + F_{j-1}\mathcal{A}\} = [a, b] + F_{i+j-d-1}\mathcal{A}$.

Examples

1) $A = \mathbb{C}[x, y]$, $\mathcal{A} = T(x, y)/(xy - yx - 1) = \mathcal{D}(\mathbb{A}^1)$.
2) $A = S(\mathfrak{g})$, $\mathcal{A} = U(\mathfrak{g})$.

Quantizations of $\mathbb{C}[\mathcal{N}]$

$\mathcal{N} \subset \mathfrak{g}^*$ – the nilpotent cone.

Theorem (Losev):

Quantizations of $\mathbb{C}[\mathcal{N}]$ are in bijection with $\mathfrak{h}^*/W$.

$\chi \in \mathfrak{h}^*/W \mapsto \mathfrak{m}_\chi \subset \mathbb{C}[\mathfrak{h}^*]^W \simeq Z(U(\mathfrak{g}))$.
$I_\chi = (m_\chi) \subset U(\mathfrak{g})$.
$\mathcal{A}_\chi = U(\mathfrak{g})/I_\chi$.
Examples show that some interesting unitary representations of G are associated with G-equivariant covers of nilpotent orbits rather than orbits themselves. Thus, we want to have a description of the set of quantizations of $\mathbb{C}[\widehat{\mathcal{O}}]$ for any orbit $\mathcal{O} \subset \mathcal{N}$, and any G-equivariant covering $\widehat{\mathcal{O}}$ of the orbit $\mathcal{O}$.

Affine conical symplectic singularities

Let X be a normal Poisson variety, and assume that the regular locus X^{reg} admits a symplectic form ω^{reg}. Following Beauville, we say that X has symplectic singularities if X admits a projective resolution of singularities $\rho : \widetilde{X} \to X$, such that $\rho^*(\omega^{reg})$ extends to a regular (not necessarily symplectic) form on $\widetilde{X}$.
We say that an affine symplectic singularity X is conical if $\mathbb{C}[X]$ is a conical Poisson algebra.

Examples of affine symplectic singularities:

1) Kleinian singularity $\mathbb{C}^2/\Gamma$, where $\Gamma \in Sp_2$ is a finite subgroup;
2) $\mathrm{Spec}(\mathbb{C}[\mathcal{O}])$ for any nilpotent orbit $\mathcal{O} \subset \mathcal{N}$;
2) $\mathrm{Spec}(\mathbb{C}[\widehat{\mathcal{O}}])$ for any G-equivariant cover $\widehat{\mathcal{O}}$ of $\mathcal{O}$.

Quantizations of affine conical symplectic singularities

Let X be an affine conical symplectic singularity.

Theorem: (Losev)

Quantizations of X are in bijection with $\mathfrak{P}/W$, where $\mathfrak{P}$ is an affine space, and W is a finite group acting on $\mathfrak{P}$ by reflections.

Examples:

1) For $X = \mathcal{N}$ we have $\mathfrak{P} = \mathfrak{h}^*$ and W is the Weyl group of $\mathfrak{g}$;
2) We can obtain a similar representation-theoretic description of $\mathfrak{P}$ and W for any $X = \mathrm{Spec}(\mathbb{C}[\widehat{\mathcal{O}}])$. Namely, there is a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, such that $\mathfrak{P} \simeq (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^*$. The description of W is more subtle.

Canonical quantizations

Let X be an affine conical symplectic singularity, and set $A = \mathbb{C}[X]$.

There is a distinguished quantization $\mathcal{A}$ of A called the canonical quantization, satisfying the following properties.

$\mathcal{A}$ is an even quantization, i.e. it admits a filtered anti-involution $\sigma : \mathcal{A} \to \mathcal{A}$, such that $\operatorname{gr} \sigma : A \to A$ sends $a \in A_i$ to $\zeta^i a$, where ζ is a primitive $2d$-th root of unity;

The action of the group of Poisson automorphisms of X on A lifts to an action on $\mathcal{A}$.

For $X = \operatorname{Spec}(\mathbb{C}[\widehat{\mathcal{O}}])$ we have an additional properties of $\mathcal{A}$.

G acts on $\mathcal{A}$, and the action admits a unique quantum comoment map $\Phi : U(\mathfrak{g}) \to \mathcal{A}$.

$\mathcal{A}$ has a structure of a Harish-Chandra bimodule over G.

We define the ideal $I(\widehat{\mathcal{O}}) \subset U(\mathfrak{g})$ to be the kernel of Φ.

Unipotent Harish-Chandra bimodules

Set $\mathcal{A}$ to be the canonical quantization of $\mathbb{C}[\widehat{\mathcal{O}}]$, and let Π be the Galois group of the covering $\widehat{\mathcal{O}} \to \mathcal{O}$. The action of Π on $\mathbb{C}[\widehat{\mathcal{O}}]$ lifts to the action on $\mathcal{A}$.

For any irreducible representation V of Π set $X_V = (\mathcal{A}_0(\widehat{\mathcal{O}}) \otimes V)^{\Pi}$.

Definition: (Losev, Mason-Brown, M.)

We define the set $Unip_{\widehat{\mathcal{O}}}(\mathcal{O})$ of unipotent Harish-Chandra bimodules associated with $\widehat{\mathcal{O}}$ to be the set $\{X_V\}$ for all irreducible representations V of Π.

We set $Unip(\mathcal{O}) = \bigcup_{\widehat{\mathcal{O}}} Unip_{\widehat{\mathcal{O}}}(\mathcal{O})$ to be the set of unipotent Harish-Chandra bimodules corresponding to the orbit $\mathcal{O}$.

Unipotent Harish-Chandra bimodules

We have the following properties of unipotent Harish-Chandra bimodules.

1) X_V is irreducible Harish-Chandra bimodule for any irreducible representation V of Π.
2) $\mathrm{LAnn}(X_V) = \mathrm{RAnn}(X_V) = I(\widehat{\mathcal{O}})$.

Proposition: (Losev, Mason-Brown, M.)

Suppose G is a classical linear group. Let $\mathcal{A}$ be the canonical quantization of $\mathbb{C}[\widehat{\mathcal{O}}]$. The ideal $I(\widehat{\mathcal{O}}) \subset U(\mathfrak{g})$ is maximal.

Proof is based on combinatorial computations, and we expect the proposition to hold for all simple G.

Example of SL_2

Set $G = SL_2$, and $\mathcal{O}$ to be the regular nilpotent orbit in $\mathfrak{sl}_2$.
For a trivial cover $\mathcal{O}$ we have the unique unipotent Harish-Chandra bimodule in $Unip_{\mathcal{O}}(\mathcal{O})$ that is the canonical quantization of $\mathbb{C}[\mathcal{O}] = \mathbb{C}[\mathcal{N}]$. In fact, this quantization is $\mathrm{Ind}_T^G \mathbb{C}$ and coincides with the special unipotent Harish-Chandra bimodule for $\mathcal{O}$.
Consider the universal 2-fold cover $\widehat{\mathcal{O}} = \mathbb{C}^2 \backslash \{0\}$. We have $\mathbb{C}[\widehat{\mathcal{O}}] = \mathbb{C}[x, y]$, and the canonical quantization is $\mathcal{A} = \mathcal{D}(\mathbb{A}^1) = \mathbb{C}[x, \frac{d}{dx}]/(\frac{d}{dx}x - x\frac{d}{dx} - 1)$. The group $\Pi = \mathbb{Z}_2$ acts on $\mathcal{A}$ by sending x to $-x$ and $\frac{d}{dx}$ to $-\frac{d}{dx}$. We have two unipotent Harish-Chandra bimodules:

$$V^{even} = \mathcal{A}^{\mathbb{Z}_2};$$

$$V^{odd} = (\mathcal{A} \otimes \mathrm{sign})^{\mathbb{Z}_2}.$$

Vogan's desiderata

1) Vogan's conjecture.
   Follows directly from the definition of a unipotent Harish-Chandra bimodule. Indeed, consider $X_V \in Unip_{\widehat{\mathcal{O}}}(\mathcal{O})$ for some representation V of Π. Set $p : \widehat{\mathcal{O}} \to \mathcal{O}$ be the covering map, and $M = p_*(\mathcal{S}_{\widehat{\mathcal{O}}} \otimes V)^{\Pi}$. Such M satisfies the condition of Vogan's conjecture.
2) For any $X \in Unip(G)$, $\mathrm{LAnn}_{U(\mathfrak{g})}(X) = \mathrm{RAnn}_{U(\mathfrak{g})}(X)$ is a maximal ideal
   Proved for classical linear group G, expected to be true for all G.
3) $Unip(\mathfrak{g}) \subset \widehat{G}$
   Proved for classical linear group G, expected to be true for all G.
4) $Unip(\mathcal{O}) \supset Unip^s(\mathcal{O})$
   Proved for classical linear group G, expected to be true for all G.

Questions to be answered:

1) Why do we have $Unip(\mathcal{O}) \supset Unip^s(\mathcal{O})$?
2) How many irreducible representations are annihilated by the ideal $I(\widehat{\mathcal{O}})$?
3) Why are unipotent representations unitary?

Generalized duality

Assume G is a classical linear group. Let $SpCov(\mathfrak{g})$ be the set of G-equivariant covers of special orbits in $\mathfrak{g}$.

Theorem: (Losev, Mason-Brown, M.)

There is an injective map $\tilde{d}: \mathcal{N}^\vee/G^\vee \to SpCov(\mathfrak{g})$, such that

- $\tilde{d}(\mathcal{O}^\vee)$ is a G-equivariant cover of $d(\mathcal{O}^\vee)$;
- $I(\tilde{d}(\mathcal{O}^\vee)) = I(\frac{1}{2}h^\vee)$.

Corollary:

$Unip(\mathcal{O}) \supset Unip^s(\mathcal{O})$.

Almost etale covers

For a G-equivariant cover $\widehat{\mathcal{O}}$ consider the ideal $I(\widehat{\mathcal{O}})$.

Proposition:

There is a unique maximal G-equivariant cover $\widetilde{\mathcal{O}}$ of $\widehat{\mathcal{O}}$ with the Galois group Π of the covering $\widetilde{\mathcal{O}} \to \mathcal{O}$, satisfying the following properties.

- $I(\widehat{\mathcal{O}}) = I(\widetilde{\mathcal{O}})$, and therefore $Unip_{\widehat{\mathcal{O}}}(\mathcal{O}) = Unip_{\widetilde{\mathcal{O}}}(\mathcal{O})$;
- X_V is not isomorphic to X_W for two non-isomorphic irreducible representations V and W of Π;
- Any irreducible Harish-Chandra bimodule X with $\mathrm{LAnn}(X) = \mathrm{RAnn}(X) = I(\widetilde{\mathcal{O}})$ is isomorphic to X_V for some irreducible representation V of Π.

Lusztig-Spaltenstein induction

Δ – the set of simple roots of $\mathfrak{g}$

$\Delta = (e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n)$

Φ – the root system of $\mathfrak{g}$ $\quad \Phi = (\{e_i - e_j\})$

$I \subset \Delta$ $\quad I = \Delta / \{e_{k-1} - e_k\}$

$\Phi_I \subset \Phi$ $\quad \Phi_I = (\{e_i - e_j | i, j \leqslant k \text{ or } i, j > k\})$

$\mathfrak{l}_I = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$ – Levi subalgebra of $\mathfrak{g}$ $\quad \mathfrak{l}_I = \mathfrak{s}(\mathfrak{gl}_k \times \mathfrak{gl}_{n-k})$;

$\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{n}_I$ – a parabolic subalgebra

$P \subset G$, $L \subset G$ – corresponding subgroups.

$\mathcal{O}_L \subset \mathfrak{l}$ – a nilpotent L-orbit.

Lusztig-Spaltenstein induction.

The image of the map $\rho : G \times^P (\overline{\mathcal{O}}_L \times \mathfrak{n}) \to \mathfrak{g}$ contains the unique open dense orbit $\mathcal{O}$.
Such orbit $\mathcal{O}$ is called induced from $(\mathcal{O}_L, \mathfrak{l})$. If orbit $\mathcal{O}$ cannot be induced from any proper Levi subalgebra $\mathfrak{l}$, we say that $\mathcal{O}$ is a rigid orbit.

Birational Lusztig-Spaltenstein induction

Let $\mathcal{O}$ be induced from $\mathcal{O}_L \subset \mathfrak{l}$.

Let $\widehat{\mathcal{O}}_L$ be an L-equivariant covering of $\mathcal{O}_L$.

$\rho : G \times^P (\mathrm{Spec}(\mathbb{C}[\widehat{\mathcal{O}}_0]) \times \mathfrak{n}) \to \mathfrak{g}$.

$\widehat{\mathcal{O}} = \rho^{-1}(\mathcal{O})$.

$\widehat{\mathcal{O}}$ is a G-equivariant covering of $\mathcal{O}$. We say that $\widehat{\mathcal{O}}$ is birationally induced from $(\widehat{\mathcal{O}}_L, \mathfrak{l})$. If $\widehat{\mathcal{O}}$ cannot be birationally induced from any proper Levi subalgebra $\mathfrak{l}$, we say that $\widehat{\mathcal{O}}$ is a birationally rigid cover.

For any covering $\widehat{\mathcal{O}}$ there is a unique pair $(\widehat{\mathcal{O}}_L, \mathfrak{l})$, such that

$\widehat{\mathcal{O}}$ is birationally induced from $\widehat{\mathcal{O}}_L$;

$\widehat{\mathcal{O}}_L$ is a birationally rigid cover.

Quantum Hamiltonian reduction

Note that $G \times^P (\mathrm{Spec}(\mathbb{C}[\widehat{\mathcal{O}}_0]) \times \mathfrak{n}) = (T^*G \times \mathrm{Spec}(\mathbb{C}[\widehat{\mathcal{O}}_0]))/\!\!/\!\!/P$ is obtained by Hamiltonian reduction.

We can use quantum Hamiltonian reduction to define parabolic induction of quantizations.

Proposition:

Suppose that $\widehat{\mathcal{O}}$ is birationally induced from $(\mathfrak{l}, \widehat{\mathcal{O}}_L)$. Then the canonical quantization of $\mathbb{C}[\widehat{\mathcal{O}}]$ is parabolically induced from the canonical quantization of $\mathbb{C}[\widehat{\mathcal{O}}_L]$.

Proposition:

Assume that $\mathcal{O}$ is not birationally rigid orbit, and let $\mathcal{O}_L \subset \mathfrak{l}^*$ be the birationally rigid orbit, such that $\mathcal{O}$ is birationally induced from $(\mathfrak{l}, \mathcal{O}_L)$. Then all $X \in Unip(\mathcal{O})$ are obtained from $Unip(\mathcal{O}_L)$ by taking (possibly twisted) parabolic induction and taking isotypic components with respect to the finite group actions.

Unitarity of unipotent representations

Proposition (Barbasch):

If $\mathcal{O}$ is a rigid orbit, then any $X \in Unip(\mathcal{O})$ is unitarizable.

It is easy to imply the analogous statement for a birationally rigid $\mathcal{O}$.
For G classical linear group the operations described in the previous slide send unitarizable Harish-Chandra bimodules to unitarizable Harish-Chandra bimodules. We expect it to be true for general G.
That implies that $Unip(G) \subset \widehat{G}$.