RESEARCH STATEMENT

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My main area of interest is the field of geometric representation theory. I am working under the supervision of Ivan Losev (Yale University), and most of my work is centered around the following two topics.

1) One-dimensional representations of finite W-algebras.
2) Unipotent representations of complex reductive groups.

The former set is closely connected to the set of irreducible representations of $\mathfrak{g}$, while studying the latter set is a key step for understanding the structure of the set of unitary representations of G.

To elaborate, let X be an irreducible finite-dimensional representation of a finite W-algebra $\mathcal{W}$, and let $J = \mathrm{Ann}_{\mathcal{W}}(X) \subset \mathcal{W}$ be the corresponding primitive two-sided ideal. In [Los10b] Losev constructed a map $\bullet^{\dagger} : \mathfrak{Id}^{\mathrm{fin}}(\mathcal{W}) \to \mathfrak{Id}(U(\mathfrak{g}))$ between the set of primitive ideals of finite codimension in $\mathcal{W}$ and the set of primitive ideals in the universal enveloping $U(\mathfrak{g})$. Thus, to a finite-dimensional irreducible representation X of $\mathcal{W}$ with annihilator $J \subset \mathcal{W}$ one can assign an irreducible representation of $U(\mathfrak{g})$, namely $U(\mathfrak{g})/J^{\dagger}$. It was shown in [Los11b] that every primitive ideal with the associated variety $\overline{\mathbb{O}}$ is of the form $\mathrm{Ann}(X)^{\dagger}$ for a finite-dimensional representation X of $\mathcal{W}$. An important first task is to understand the structure of 1-dimensional representation X of $\mathcal{W}$. That is a joint ongoing project with Ivan Losev, see Section 1 and Section 3.1 for more details.

The set of unipotent representations is a finite set of unitary representations that are assigned to nilpotent orbits in $\mathfrak{g}$ (or, in general, G-equivariant covers of nilpotent orbits). It is expected that all unitary representations can be constructed from the set of unipotent ones through several types of induction. In [LMM21] together with Ivan Losev and Lucas Mason-Brown we propose a new definition of unipotent representations of a complex reductive group G, prove that it satisfies many of the desired properties, and give a classification of the set of unipotent representations, see Section 2 for more details.

In Section 3 I describe the three projects I am currently involved in. First one is above mentioned project with Ivan Losev towards the desciption of the structure of the set of 1-dimensional representations of finite W-algebras. The second is a joint project with Ivan Losev and Lucas Mason-Brown, in which we work towards the definition of a unipotent Harish-Chandra module. The third project analyzes the connection between the extended Barbasch-Vogan-Lusztig-Spaltenstein duality proposed in [LMM21] and the symplectic duality.

1. One-dimensional representations of a finite W-algebra

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit. To a nilpotent element $e \in \mathbb{O}$ one can assign a finite W-algebra $\mathcal{W} = U(\mathfrak{g}, e)$, see [Pre02]. We note that, up to isomorphism, $\mathcal{W}$ depends only on the orbit $\mathbb{O}$, and not the element $e \in \mathbb{O}$. There is a distinguished transversal slice S to $\mathbb{O}$ at e, called the Slodowy slice, and $\mathcal{W}$ is a quantization of the algebra of functions $\mathbb{C}[S]$. The goal of a joint project with Ivan Losev is to classify all 1-dimensional representations of $\mathcal{W}$ in classical types.

Let G be the adjoint algebraic group corresponding to $\mathfrak{g}$, and let $Q \subset G$ be the reductive part of the stabilizer of the point $e \in \mathfrak{g}$. Then there is a natural action of Q on $\mathcal{W}$ and therefore on the set $\mathrm{Rep}_1(\mathcal{W})$ of 1-dimensional representations of $\mathcal{W}$. Moreover, the action of Q on the latter is factored through $\Gamma := Q/Q^\circ = \pi_1^G(\mathbb{O})$.

1.1. **Γ-stable one-dimensional representations of a finite W-algebra.** In [PT14], Premet and Topley proved that for a classical simple Lie algebra $\mathfrak{g}$ the set $\mathrm{Rep}_1^\Gamma(\mathcal{W})$ is an affine space. Their approach was completely algebraic in nature. In [Mat18] I provide an alternative geometric approach to the problem and give another proof of this fact.

By the work of Losev [Los10a], the set of Γ-stable 1-dimensional representations of $\mathcal{W}$ is in a bijection with the set of quantizations of the nilpotent orbit $\mathbb{O}$ considered as a symplectic variety with Konstant-Kirillov symplectic form. The key result of [Mat18] is the following one.

Theorem 1.1.1. *Let $\mathfrak{g}$ be a classical simple Lie algebra. Then every quantization of $\mathbb{O}$ can be uniquely extended to a quantization of* $\mathrm{Spec}(\mathbb{C}[\mathbb{O}])$.

The argument of the proof works for any nilpotent orbit $\mathbb{O}$, such that for any codimension 2 orbit $\mathbb{O}' \subset \overline{\mathbb{O}}$ the corresponding singularity is either a Kleinian singularity or a union of Kleinian singularities transversally meeting at the singular point. By [KP82], that is always true for classical types. In [FJLS15] it was shown that in exceptional types there are non-normal codimension 2 singularities in $\overline{\mathbb{O}}$ with the normalization isomorphic to $\mathbb{C}^2$ and $\mathbb{C}^2/\mathbb{Z}_4$ respectively. In this case, the statement of Theorem 1.1.1 is false.

The set of quantizations of $\mathrm{Spec}(\mathbb{C}[\mathbb{O}])$, or equivalently of the algebra $\mathbb{C}[\mathbb{O}]$, is known due to the work of Losev [Los15] and is in a bijection with the set of points of an affine space, thus deducing the analogous statements for the set $\mathrm{Rep}_1^\Gamma(\mathcal{W})$. Moreover, results of loc.cit. imply that all quantizations of $\mathrm{Spec}(\mathbb{C}[\mathbb{O}])$ are obtained by a parabolic induction, mirroring the parabolic induction on the respective sets of 1-dimensional representations, see [Los11a] for details.

For a Levi subgroup $L \subset G$ choose a parabolic $P \subset G$ containing L, and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ be the corresponding parabolic subalgebra. Recall that the orbit $\mathbb{O}$ is obtained by *Lusztig-Spaltenstein* induction from the pair $(L, \mathbb{O}_L)$ consisting of an Levi subgroup $L \subset G$, and a nilpotent orbit $\mathbb{O}_L \subset \mathfrak{l}$ if $\mathbb{O}$ is the unique open G-orbit in the image of the generalized Springer map $\rho : G \times^P (\overline{\mathbb{O}}_L \times \mathfrak{n}) \to \mathfrak{g}$. We say that $\mathbb{O}$ is *birationally induced* from $(L, \mathbb{O}_L)$ if moreover the map ρ is birational, and that $\mathbb{O}$ is *birationally rigid* if it cannot be birationally induced from any proper Levi L. The following is the second main result of [Mat18].

Theorem 1.1.2. *Let $\mathfrak{g}$ be a classical simple Lie algebra, and $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit. Suppose that $\mathbb{O}$ is birationally induced from $(L, \mathbb{O}_L)$, and $\mathbb{O}_L$ is birationally rigid. Then all Γ-stable 1-dimensional representations of $\mathcal{W}$ are parabolically induced from $\mathfrak{l}$.*

1.2. **Singularities of** $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$**.** Let $\widetilde{\mathbb{O}}$ be a G-equivariant cover of $\mathbb{O}$. We would like to prove a similar to Theorem 1.2.2 result for $\widetilde{\mathbb{O}}$. In order to extend a quantization from $\widetilde{\mathbb{O}}$ to $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$, one needs first to understand the codimension 2 symplectic leaves of $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$, and the respective singularities. That is the main goal of this section, we refer to Section 3.1 for the next steps towards the generalization of Theorem 1.2.2.

Recall first that $\mathrm{Spec}(\mathbb{C}[\mathbb{O}])$ is the normalization of the closure $\overline{\mathbb{O}}$. The codimension 2 leaves in $\mathrm{Spec}(\mathbb{C}[\mathbb{O}])$ are the codimension 2 G-orbits, and the singularities are easily determined from the corresponding singularities in $\overline{\mathbb{O}}$. The latter are well-known due to the work of Kraft an Procesi [KP82].

Theorem 1.2.1. *[KP82] Let $\mathbb{O}' \subset \overline{\mathbb{O}}$ be a codimension 2 orbit. Then the singularity of $\mathbb{O}' \subset \overline{\mathbb{O}}$ is equivalent to one of the following.*

a) Kleinian singularity of type A_1;
b) Kleinian singularity of type D_{k+1};
c) Kleinian singularity of type A_{2k-1};
d) Union of two Kleinian singularities of type A_{2k-1} transversally meeting at 0.

In [Mat20] I computed the corresponding singularities for $\mathrm{Spec}(\mathbb{C}[\widehat{\mathbb{O}}])$, where $\widehat{\mathbb{O}}$ is the universal G-equivariant cover of $\mathbb{O}$, and G is SO_n or Sp_{2n}. Namely, we have the following.

Theorem 1.2.2. *Let $\mathbb{O}' \subset \overline{\mathbb{O}}$ be a codimension 2 orbit. The preimage $\widetilde{\mathbb{O}}'$ of $\mathbb{O}'$ under the map $\mathrm{Spec}(\mathbb{C}[\widehat{\mathbb{O}}]) \to \overline{\mathbb{O}}$ is connected, and the singularity of $\widetilde{\mathbb{O}}'$ in $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$ is equivalent to one of the following.*

a) Either Kleinian singularity of type A_1 or $\widetilde{\mathbb{O}}' \subset \mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])^{reg}$, depending on the partitions corresponding to $\mathbb{O}$ and $\mathbb{O}'$;
b) Either Kleinian singularity of type D_{k+1} or a Kleinian singularity of type A_{2k-3} depending on the partitions corresponding to $\mathbb{O}$ and $\mathbb{O}'$;
c) Kleinian singularity of type A_{2k-1};
d) Kleinian singularity of type A_{2k-1}.

The exact conditions for the partitions in cases a) and b) are given in [Mat20, Theorem 2.6]. To simplify the exposition we won't provide them there. Note that using methods described in the paper, one can compute the singularities of $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$ for any G-equivariant cover $\widetilde{\mathbb{O}}$ of $\mathbb{O}$.

2. Unipotent representations of complex reductive groups

Let G be a real reductive group. It is conjectured that all unitary representations of G are constructed through induction from a finite set of irreducible unitary representations associated with nilpotent orbits called *unipotent* representations. There are several proposed definitions of unipotent representations, most notably the set of special unipotent representations, see [Art83] and [BV85]. However, it is clear that there are many interesting representations that should be considered unipotent but are not special unipotent, such as, e.g. the oscillator representations of $Mp(2n, \mathbb{R})$. We also note that the set of special unipotent representations is too small to generate all unitary representations via induction. There were several attempts to give a more general definition of the set of unipotent representations, see [Vog84], [Bar89], [McG94] and [Bry03]. Some of these approaches are case-dependent, and based on ad hoc definitions. Moreover, they either include non-unitary representations or miss some of the interesting representations we would like to include. In a joint monograph [LMM21] with Ivan Losev and Lucas Mason-Brown we propose a new definition of unipotent representations for complex reductive groups and show that it satisfies many of the expected properties.

2.1. **Definitions and properties of unipotent ideals.** Let G be a complex reductive group, $\mathfrak{g}$ be the Lie algebra of G, and $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit. Consider a G-equivariant cover $\widetilde{\mathbb{O}}$ of $\mathbb{O}$. The set of quantizations of $\mathbb{C}[\widetilde{\mathbb{O}}]$ is in a bijection with the set of points of an affine space $\mathfrak{z}(\mathfrak{l})/W$ for some Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, and some finite group W. There is a distinguished W-stable point $0 \in \mathfrak{z}(\mathfrak{l})$, and let $\mathcal{A}_0(\widetilde{\mathbb{O}})$ be the quantization of $\mathbb{C}[\widetilde{\mathbb{O}}]$ corresponding to the parameter $0 \in \mathfrak{z}(\mathfrak{l})/W$. We call $\mathcal{A}_0(\widetilde{\mathbb{O}})$ the *canonical* quantization of $\mathbb{C}[\widetilde{\mathbb{O}}]$. The action of G on $\mathbb{C}[\widetilde{\mathbb{O}}]$ lifts to an action on $\mathcal{A}_0(\widetilde{\mathbb{O}})$, and it admits a quantum comoment map $\Phi : U(\mathfrak{g}) \to \mathcal{A}_0(\widetilde{\mathbb{O}})$. We set $I_0(\widetilde{\mathbb{O}})$ to be the kernel of Φ, and say that $I_0(\widetilde{\mathbb{O}})$ is the *unipotent* ideal corresponding to the cover $\widetilde{\mathbb{O}}$. The following theorem is proved for linear classical groups in [LMM21] and extended to all reductive groups in a joint paper [MM21] with Mason-Brown.

Theorem 2.1.1. *Let $\widetilde{\mathbb{O}}$ be a G-equivariant cover of a nilpotent orbit. The associated unipotent ideal $I_0(\widetilde{\mathbb{O}})$ is maximal.*

For a classical simple group G, the infinitesimal characters of the unipotent ideals are computed in [LMM21]. For Spin and exceptional groups the computations are carried in [MM21]. Many of these ideals were not studied before. It is important to note that the set of unipotent ideals includes the set of special unipotent ideals. We follow Barbasch and Vogan [BV85] to define the latter set below.

Let $\mathfrak{g}^\vee$ be the Langlands dual Lie algebra, and $\mathbb{O}^\vee \subset \mathfrak{g}^\vee$ be a nilpotent orbit. Fix Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee$ respectively. Pick an $\mathfrak{sl}_2$-triple $(e^\vee, f^\vee, h^\vee)$ with $e^\vee \in \mathbb{O}^\vee$, such that $h^\vee \in \mathfrak{h}^\vee$, and set $\gamma_{\mathbb{O}^\vee} = \frac{1}{2}h^\vee \in \mathfrak{h}^\vee \simeq \mathfrak{h}^*$. The unique maximal ideal $I(\gamma_{\mathbb{O}^\vee})$ with the central character $\gamma_{\mathbb{O}^\vee}$ is called the *special unipotent* ideal corresponding to the orbit $\mathbb{O}^\vee$. It is well-known that the associated variety of $I(\gamma_{\mathbb{O}^\vee})$ is the closure of a special nilpotent orbit $\mathbb{O} = \mathsf{D}(\mathbb{O}^\vee)$, where $\mathsf{D} : \{\text{nilpotent orbits in } (\mathfrak{g}^\vee)\} \to \{\text{nilpotent orbits in } \mathfrak{g}\}$ is the Barbash-Vogan-Lusztig-Spaltenstein duality. We proved the following:

Theorem 2.1.2. *There is an injective map*

$$\widetilde{\mathsf{D}} : \{\textit{nilpotent orbits in } (\mathfrak{g}^\vee)\} \hookrightarrow \{\textit{covers of nilpotent orbits in } \mathfrak{g}\}$$

that we call extended Barbasch-Vogan-Lusztig-Spaltenstein duality *satisfying the following properties:*

- *For every nilpotent orbit $\mathbb{O}^\vee \subset \mathfrak{g}^\vee$, $\widetilde{\mathsf{D}}$ is a connected G-equivariant cover of $\mathsf{D}(\mathbb{O}^\vee)$;*
- *The central character of the unipotent ideal $I_0(\widetilde{\mathsf{D}}(\mathbb{O}^\vee))$ equals to $\gamma_{\mathbb{O}^\vee}$.*

As an immediate corollary, every special unipotent ideal is unipotent. The construction of the map $\widetilde{\mathsf{D}}$ is expected to be a special case of *symplectic duality*, see Section 3.3 for more details.

2.2. **Unipotent Harish-Chandra bimodules.** Let G be a complex reductive group, and let $\widetilde{\mathbb{O}}$ be a G-equivariant cover of a nilpotent orbit $\mathbb{O}$. Let $\mathcal{B}$ be a $U(\mathfrak{g})$-bimodule endowed with an ascending filtration

$$0 = \mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \dots, \quad \bigcup_i \mathcal{B}_i = \mathcal{B}$$

compatible with the filtration on $U(\mathfrak{g})$ such that

$$U_i(\mathfrak{g})\mathcal{B}_j \subset \mathcal{B}_{i+j}, \quad [U_i(\mathfrak{g}), \mathcal{B}_j] \subset \mathcal{B}_{i+j-1}.$$

Then $\operatorname{gr}\mathcal{B}$ has a structure of a graded $S(\mathfrak{g})$-module. Such filtration is called good if $\operatorname{gr}\mathcal{B}$ is a finitely generated $S(\mathfrak{g})$-module, and a $U(\mathfrak{g})$ bimodule is called Harish-Chandra if it admits a good filtration.

We say that an irreducible Harish-Chandra bimodule $\mathcal{B}$ is a *unipotent* Harish-Chandra bimodule associated with $\widetilde{\mathbb{O}}$ if left and right annihilators of $\mathcal{B}$ are equal to $I_0(\widetilde{\mathbb{O}})$. We denote the set of unipotent Harish-Chandra bimodules associated with $\widetilde{\mathbb{O}}$ by $\operatorname{Unip}_{\widetilde{\mathbb{O}}}(G)$. Define the set $\operatorname{Unip}(G)$ of unipotent Harish-Chandra bimodules to be the union of $\operatorname{Unip}_{\widetilde{\mathbb{O}}}(G)$ for all covers $\widetilde{\mathbb{O}}$. Similarly, say that $\mathcal{B}$ is a special unipotent Harish-Chandra bimodule if both left and right annihilators of $\mathcal{B}$ are equal to the same special unipotent ideal. We note that due to Theorem 2.1.2, every special unipotent bimodule is unipotent. Below we catalogue the main properties and the classification of unipotent Harish-Chandra bimodules. The reference for all these statements is [LMM21].

Theorem 2.2.1. *Assume that G is a linear classical group, and $\widetilde{\mathbb{O}}$ is a G-equivariant cover of a nilpotent orbit. Then all bimodules in $\operatorname{Unip}_{\widetilde{\mathbb{O}}}(G)$ are unitarizable.*

Similarly to the analogous result of [BV85] for the special unipotent bimodules, the unipotent bimodules annihilated by the unipotent ideal $I_0(\widetilde{\mathbb{O}})$ are parameterized by the irreducible representations of a finite group. Moreover, there is a geometric description of this group. Namely, we have the following result.

Theorem 2.2.2. *Let I be a unipotent ideal. There is a unique maximal G-equivariant cover $\widetilde{\mathbb{O}}$ of a nilpotent orbit, such that $I_0(\widetilde{\mathbb{O}}) \simeq I$. Set Π to be the automorphism group of the covering $\widetilde{\mathbb{O}} \to \mathbb{O}$. There is a natural bijection between the set $\operatorname{Unip}_{\widetilde{\mathbb{O}}}(G)$ and the set of irreducible representations of Π. Moreover, if V is the finite-dimensional representation of Π, corresponding to $\mathcal{B} \in \operatorname{Unip}_{\widetilde{\mathbb{O}}}(G)$ we have an isomorphism of graded G-equivariant $S(\mathfrak{g})$-modules*

$$\operatorname{gr}(\mathcal{B}) \simeq (\mathbb{C}[\widetilde{\mathbb{O}}] \otimes V)^{\Pi}.$$

Fix a point $e \in \mathbb{O}$. In [Vog91], Vogan conjectured that for any unipotent Harish-Chandra bimodule $\mathcal{B}$ there is a finite-dimensional representation χ of $\pi_1^G(\mathbb{O}) = G_e/G_e^\circ$, such that there is an isomorphism of G-representations $\mathcal{B} \simeq_G \operatorname{AlgInd}_{G_e}^G \chi$. We note that this conjecture can be implied from Theorem 2.2.2.

3. Future plans

3.1. **Towards the description of** $\operatorname{Rep}_1(\mathcal{W})$**.** Let $\Omega \subset \Gamma = \pi_1^G(\mathbb{O})$ be a subgroup, and let $\widetilde{\mathbb{O}} \to \mathbb{O}$ be the corresponding G-equivariant covering. As mentioned in Section 1.1, a result of [Los10a] states that the set $\operatorname{Rep}_1^\Omega(\mathcal{W})$ of Ω-stable 1-dimensional representations of $\mathcal{W}$ is in a bijection with the set of quantizations of $\widetilde{\mathbb{O}}$. The following statement proposed by Ivan Losev is the key conjecture for the study of $\operatorname{Rep}_1(\mathcal{W})$.

Conjecture 3.1.1. *Let X be a 1-dimensional representation of $\mathcal{W}$, $\widetilde{\mathbb{O}}$ be a G-equivariant cover of $\mathbb{O}$, and let $\mathcal{D}$ be a quantization of $\widetilde{\mathbb{O}}$ corresponding to X. Then one of the following is true*

- $\pi_1(\widetilde{\mathbb{O}}) \subsetneq \operatorname{Stab}_\Gamma(X)$;
- *$\mathcal{D}$ can be extended to a quantization of $\operatorname{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$, i.e. $\Gamma(\mathbb{O}, \mathcal{D})$ is a quantization of $\mathbb{C}[\widetilde{\mathbb{O}}]$.*

We note that for $\widetilde{\mathbb{O}} = \mathbb{O}$ the statement of the conjecture coincides with Theorem 1.1.1.

For $\mathfrak{g}$ of type A, the conjecture is well-known and easy to prove. For the exceptional $\mathfrak{g}$ the conjecture is false, see [Pre13]. In an ongoing project with Ivan Losev we focus on proving the conjecture for $G = SO_n$ or $G = Sp_{2n}$. It is important to note that the conjecture for classical types follows from the recent preprint [Top21] of Lewis Topley, where it is proved using completely different methods. However, we believe that the methods that we develop for this project can lead to additional results and can be applicable to many orbits in exceptional types. We also expect that some of the results can be applied to more generic symplectic singularities. Let us elaborate on the main ideas and results of this project.

The key part of proving Conjecture 3.1.1 is extending $\mathcal{D}$ to the codimension 2 G-orbits in $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$. First, we need to understand the corresponding singularities, that is the main result of [Mat20], see Section 1.2 for details. Second, we analyze the restriction of $\mathcal{A} = \Gamma(\widetilde{\mathbb{O}}, \mathcal{D})$ to slices to codimension 2 orbits in $\overline{\mathbb{O}}$.

Namely, let $\mathbb{O}' \subset \overline{\mathbb{O}}$ be a codimension 2 orbit, and let Σ be the Kleinian singularity corresponding to (an irreducible component of) the slice to $\mathbb{O}'$. Set $\mathcal{A}_\dagger$ to be the restriction of $\mathcal{A}$ to this slice, see [Los11b] for more details about the restriction functor $\bullet_\dagger$.

Proposition 3.1.2. *There is a quantization $\mathcal{B}$ of $\mathbb{C}[\Sigma]$, such that $\mathcal{A}_\dagger$ is a Harish-Chandra bimodule of full support over $\mathcal{B}$. Moreover, for every finite-dimensional Harish-Chandra bimodule V over $\mathcal{B}$ we have*

- $\mathrm{Hom}(V, \mathcal{A}_\dagger) = 0$;
- $\mathrm{Ext}^1(V, \mathcal{A}_\dagger) = 0$,

where the functors Hom *and* Ext^1 *are taken in the category of Harish-Chandra bimodules over $\mathcal{B}$.*

The Hom and Ext vanishing properties of $\mathcal{A}_\dagger$ show that it is completely determined by the restriction of $\mathcal{A}_\dagger$ to the regular locus of Σ, which coincides with the restriction of $\mathcal{D}$. Thanks to the classification of Harish-Chandra bimodules of full support over $\mathcal{B}$, obtained in [Los21], that allows us to describe $\mathcal{A}_\dagger$. To proceed with the idea of the proof of Conjecture 3.1.1 we need some additional notations.

The notion of birational induction given in Section 1.1 can be easily generalized to G-equivariant covers of nilpotent orbits. Namely, for a Levi subgroup $L \subset G$ and an L-equivariant cover $\widetilde{\mathbb{O}}_L$ of a nilpotent orbit $\mathbb{O}_L \subset \mathfrak{l}$, we say that $\widetilde{\mathbb{O}}$ is *birationally induced* from $(L, \widetilde{\mathbb{O}}_L)$ if $\widetilde{\mathbb{O}}$ is the open G-orbit in $G \times^P (\widetilde{\mathbb{O}}_L \times \mathfrak{n})$, where P is a parabolic subgroup containing L, and $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$. We say that a cover $\widetilde{\mathbb{O}}$ is *birationally rigid* if it cannot be birationally induced from a proper Levi subgroup. Similarly, we can talk about the quantizations of $\widetilde{\mathbb{O}}$ and $\mathbb{C}[\widetilde{\mathbb{O}}]$ being parabolically induced from a quantization of $\widetilde{\mathbb{O}}_L$ and $\mathbb{C}[\widetilde{\mathbb{O}}_L]$ respectively. Say that $\widetilde{\mathbb{O}}$ is *smoothly* induced from $(L, \widetilde{\mathbb{O}}_L)$ if $\widetilde{\mathbb{O}}$ and $\mathbb{O}$ are birationally induced from $(L, \widetilde{\mathbb{O}}_L)$ and $(L, \mathbb{O}_L)$ respectively.

The following conjecture is a key step in proving Conjecture 3.1.1.

Conjecture 3.1.3. *Set $\mathfrak{g}$ to be a classical simple Lie algebra. Let $\widetilde{\mathbb{O}}$ be a G-equivariant cover of $\mathbb{O}$, and assume that $\widetilde{\mathbb{O}}$ is smoothly induced from $(L, \widetilde{\mathbb{O}}_L)$. Then any quantization $\mathcal{D}$ of $\widetilde{\mathbb{O}}$ is parabolically induced from a quantization of $\widetilde{\mathbb{O}}_L$.*

Let X_L be an irreducible (possibly non-normal) affine variety with an open L-orbit $\widetilde{\mathbb{O}}_L$ and a finite L-equivariant map $X_L \to \overline{\mathbb{O}}_L$. Set $Y = G \times^P (X_L \times \mathfrak{n})$, and remark that we have a

G-equivarian finite map $\rho : Y \to \overline{\mathbb{O}}$. Let $\overline{\mathbb{O}}^{\mathrm{sreg}}$ be the union of $\mathbb{O}$ and all codimension 2 orbits in $\overline{\mathbb{O}}$, and set $Y^{\mathrm{sreg}} = \rho^{-1}(\overline{\mathbb{O}}^{\mathrm{sreg}})$. Below is the proposed plan of proving Conjecture 3.1.3 due to Ivan Losev.

- Use Proposition 3.1.2 to show that there is X_L as above, such that $\mathcal{D}$ extends to a quantization of Y^{sreg};
- Prove that any quantization $\mathcal{D}$ of Y^{sreg} is uniquely determined by its classifying data: restrictions of $\mathcal{D}$ to $\widetilde{\mathbb{O}}$ and the preimages of the slices to codimension 2 orbits in $\overline{\mathbb{O}}$;
- Construct a quantization $\mathcal{D}_L$ of X_L, such that the restriction of the parabolically induced quantization of Y to Y^{sreg} has the same classifying data as $\mathcal{D}$.

We note that Conjecture 3.1.3 reduces the proof of Conjecture 3.1.1 to the case of $\widetilde{\mathbb{O}}$ that cannot be smoothly induced from any proper Levi $L \subset G$. It is easy to show that in this case $\mathrm{Spec}(\mathbb{C}[\widetilde{\mathbb{O}}])$ is smooth outside of codimension 4. Such covers $\widetilde{\mathbb{O}}$ are expected to be handled separately.

3.2. **Towards the definition of a unipotent HC-module.** Let G be a semisimple complex Lie group, and $\mathfrak{g}$ be the corresponding Lie algebra. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an involution, and $\mathfrak{k} = \mathfrak{g}^{\sigma}$. Take K be a connected Lie group with a Lie algebra $\mathfrak{k}$, such that the embedding $\mathfrak{k} \to \mathfrak{g}$ lifts to either an embedding $K \hookrightarrow G$ or a group homomorphism $K \to G$ with kernel isomorphic to $\mathbb{Z}_2$. In a joint project with Ivan Losev and Lucas Mason-Brown we work towards producing a definition of a unipotent HC $(\mathfrak{g}, K)$-module. The key properties expected from unipotent HC $(\mathfrak{g}, K)$-modules are the following:

- The annihilator of a unipotent HC $(\mathfrak{g}, K)$-module is a unipotent ideal in $U(\mathfrak{g})$, defined as in Section 2.1;
- A unipotent HC $(\mathfrak{g}, K)$-module $\mathcal{B}$ is unitarizable;
- Any special unipotent HC module, i.e. a HC module annihilated by a special unipotent ideal, is unipotent.

We note that contrary to the case of HC-bimodules, not any HC $(\mathfrak{g}, K)$-module annihilated by a unipotent ideal is unitarizable, counterexamples can be found for $\mathfrak{g} = \mathfrak{sl}_2$. However, for many cases we have a good guess on which HC $(\mathfrak{g}, K)$-modules should be considered unipotent.

Fix a nilpotent orbit $\mathbb{O} \subset \mathcal{N}$, and assume that $\mathcal{B}$ is a HC $(\mathfrak{g}, K)$-module with the asociated variety $\mathsf{AV}(\mathcal{B}) = \overline{\mathbb{O}}$. In [LMM21] we stated the following conjecture, supported by a lot of computational evidence.

Conjecture 3.2.1. *Suppose that* $\mathrm{codim}_{\overline{\mathbb{O}}}(\overline{\mathbb{O}} - \mathbb{O}) \geqslant 4$. *Then every irredicble HC* $(\mathfrak{g}, K)$*-module annihilated by a unipotent ideal is unitarizable.*

We expect that the property of being unitarizable and thus unipotent for a Harish-Chandra module annihilated by a unipotent ideal is determined by the restriction to codimension 2 orbits in $\overline{\mathbb{O}}$. Let $\mathbb{O}' \subset \overline{\mathbb{O}}$ be a codimension 2 orbit, and pick a point $e \in \mathbb{O}'$. Recall the restriction functor $\bullet_{\dagger}$ for Harish-Chandra modules defined in [Los15, Section 6.1]. Similar to a discussion in Section 3.1, $\bullet_{\dagger}$ should send Harish-Chandra $(\mathfrak{g}, K)$-modules to Harish-Chandra modules over a quantization $\mathcal{B}$ of a Kleinian singularity.

We focus on the situation when the corresponding singularity is of type A_1. In many cases there is an embedding $SL_2 \hookleftarrow G_e$, such that the induced action of SL_2 on the slice to $\mathbb{O}' \subset \overline{\mathbb{O}}$ coincides with the standard SL_2 action on $\mathbb{C}^2/\{\pm 1\}$. Then $\bullet_{\dagger}$ can be modified to a functor to HC $U(\mathfrak{sl}_2)$-modules for $\mathfrak{sl}_2$. In general, it is unclear at the moment what a unipotent Harish-Chandra $\mathcal{B}$-module should be.

In [LMM21, Section 10] we describe the induction of Harish-Chandra bimodules geometrically. We expect to obtain a similar description for Harish-Chandra modules.

3.3. **Symplectic duality for the Slodowy slice.** Let X be a conical symplectic singularity, and $Y \to X$ be a $\mathbb{Q}$-terminalization. To X one can assign the following data:

- $\mathfrak{P}^X = H^2(Y^{\mathsf{reg}}, \mathbb{C})$ – the Namikawa space of X;
- W^X – the Namikawa-Weyl group of X acting on $\mathfrak{P}^X$;
- H^X – the group of graded Hamiltonian automorphisms of $\mathbb{C}[X]$;
- $\mathfrak{t}^X$ – the Lie algebra of a maximal torus $T^X \subset H^X$;
- $\mathbb{W}^X = N_{H^X}(T^X)/T^X$.

The symplectic duality is a conjectural duality between the conical symplectic singularities. It is expected that if $X^\vee$ is the symplectic dual of X, the following conditions hold:

- $(X^\vee)^\vee \simeq X$;
- $\mathfrak{P}^X \simeq \mathfrak{t}^{X^\vee}$ and $\mathfrak{P}^{X^\vee} \simeq \mathfrak{t}^X$;
- $W^X \simeq \mathbb{W}^{X^\vee}$ and $\mathbb{W}^X \simeq W^{X^\vee}$.

I plan to focus on a specific case of the symplectic duality. Let G be a semisimple complex group, and $G^\vee$ be the Langlands dual group. Let $\mathbb{O}^\vee \subset \mathcal{N}^\vee$ be a nilpotent orbit, and pick a nilpotent element $e^\vee \in \mathbb{O}^\vee$. Let $S^\vee$ to be a Slodowy slice to $e^\vee$, and set $X^\vee = S^\vee \cap \mathcal{N}^\vee$. Note that $X^\vee$ is a conical symplectic singularity. Recall from Theorem 2.1.2 the extended Barbasch-Vogan-Lusztig-Spaltenstein duality $\widetilde{\mathsf{D}} : \{\text{nilpotent orbits in } (\mathfrak{g}^\vee)\} \hookrightarrow \{\text{covers of nilpotent orbits in } \mathfrak{g}\}$, and set $X = \operatorname{Spec}(\mathbb{C}[\widetilde{\mathsf{D}}(\mathbb{O})])$.

Proposition 3.3.1. *X and $X^\vee$ are symplectic dual to each other.*

I plan to study the properties of the symplectic duality in this example. One of the main topics of interest for me is the *deformed Hikita conjecture*. Let $\mathcal{A}_{\mathfrak{P}^X,h}$ be the universal quantization of $\mathbb{C}[\widetilde{\mathsf{D}}(\mathbb{O}^\vee)]$, i.e. a $\mathbb{C}[\mathfrak{P}^X, h]$-algebra, such that all quantizations of $\mathbb{C}[\widetilde{\mathsf{D}}(\mathbb{O}^\vee)]$ are obtained by specializing to a point in $\mathfrak{P}^X$. Choose a generic torus $\nu : \mathbb{C}^\times \to T^X$. Since T^X acts on $\mathcal{A}_{\mathfrak{P}^X,h}$, ν produces a grading on $\mathcal{A}_{\mathfrak{P}^X,h}$. Construct the Cartan quotient by setting

$$C_\nu(\mathcal{A}_{\mathfrak{P}^X,h}) = \mathcal{A}^0_{\mathfrak{P}^X,h} / \sum_{i>0} \mathcal{A}^{-i}_{\mathfrak{P}^X,h} \mathcal{A}^i_{\mathfrak{P}^X,h}.$$

Let $Y^\vee$ be a $\mathbb{Q}$-terminalization of $X^\vee$. The main goal of this project is to prove for $X^\vee = S^\vee \cap \mathcal{N}^\vee$ and $X = \operatorname{Spec}(\mathbb{C}[\widetilde{\mathsf{D}}(\mathbb{O})])$ the following conjecture proposed by Nakajima, see [KTWWY19, Conjecture 8.9].

Conjecture 3.3.2. *There is a $\mathbb{C}[(\mathfrak{t}^X)^* \oplus \mathfrak{P}^X, h]$-linear isomorphism*

$$C_\nu(\mathcal{A}_{\mathfrak{P}^X,h}) \simeq H^*_{T^{X^\vee} \times \mathbb{C}^\times}(Y^\vee).$$

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