LECTURE 4.5: SOERGEL'S THEOREM AND SOERGEL BIMODULES

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Abstract. These are notes for a talk given at the MIT-Northeastern Graduate Student Seminar on category $\mathcal{O}$ and Soergel bimodules, Fall 2017.

Contents

1. Goals and structure of the talk

This is the continuation of a talk given week ago. Last week we defined Soergel's $\mathbb{V}$-functor, stated three theorems of Soergel and proved the first one. The exposition of this talk will be as follows. In Section 2 we will recall the key points of the last talk. In Section 3 and 4 we will prove the second and the third theorem. In Section 5 we define Soergel modules and bimodules. We will show that the category of Soergel modules is equivalent to the subcategory of projective objects in the principal block $\mathcal{O}_0$.

2. Reminder of last time

First, let me list notations and objects introduced in the last talk.
We set Δ_{min} to be the Verma module corresponded to the longest element in the Weyl group and P_{min} its projective cover.
We denote by C the coinvariant algebra $\mathbb{C}[\mathfrak{h}]/(\mathbb{C}[\mathfrak{h}]^W_+)$ where $\mathbb{C}[\mathfrak{h}]^W_+ \subset \mathbb{C}[\mathfrak{h}]^W$ is the ideal of all elements without constant term.
For $\lambda+\rho$, $\mu+\rho$ dominant and $W_\lambda \subset W_\mu$ we have defined the extended translation functors $\tilde{T}_{\lambda\to\mu} : \tilde{\mathcal{O}}_\lambda \to \tilde{\mathcal{O}}_\mu$ where $\tilde{\mathcal{O}}_\lambda$ is the infinitesimal block of the category $\mathcal{O}$ over $U(\mathfrak{g}) \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$ corresponding to λ. The extended translation functor $\tilde{T}_{\lambda\to\mu}$ has a left adjoint $\tilde{T}^!_{\lambda\to\mu}$ and a right adjoint $\tilde{T}^*_{\lambda\to\mu}$. We have shown that extended translation functors are transitive in the following sense: $\tilde{T}_{\lambda\to\nu} = \tilde{T}_{\mu\to\nu} \circ \tilde{T}_{\lambda\to\mu}$.

The Soergel $\mathbb{V}$-functor $\mathbb{V} : \mathcal{O}_0 \to \operatorname{End}(P_{min})^{opp}$-mod is defined by $\mathbb{V}(\bullet) = \operatorname{Hom}(P_{min}, \bullet)$.

The main result of the previous talk was the following theorem.

Theorem 2.1. $\operatorname{End}_{\mathcal{O}}(P_{min}) \simeq C$.

In the proof we have shown that for $\lambda = 0$ and $\mu = -\rho$ the extended translation functor $\tilde{T}_{\lambda\to\mu}$ coincides with the $\mathbb{V}$ functor under equivalences $\mathcal{O}_0 \simeq \tilde{\mathcal{O}}_0$ and $\tilde{\mathcal{O}}_{-\rho} \simeq C$-mod. We proved the following fact.

Proposition 2.2. *The adjunction unit map $P_{min} \to \mathbb{V}^*\mathbb{V}(P_{min})$ is an isomorphism.*

The main goal of this talk is to prove the following two theorems of Soergel.

Theorem 2.3. *$\mathbb{V}$ is fully faithful on projectives.*

Theorem 2.4. $\mathbb{V}(\mathcal{P}_i\bullet) \simeq \mathbb{V}(\bullet) \underset{\mathbb{C}[\mathfrak{h}]^{s_i}}{\otimes} \mathbb{C}[\mathfrak{h}]$.

3. $\mathbb{V}$ is fully faithful

In this section we will prove Theorem 2.3. In the proof we will need a criterion for a module to be standardly filtered.

3.1. **Criterion for a standardly filtered module.** It was shown in Daniil's talk that $\mathcal{O}_\lambda$ is a highest weight abelian category.

Proposition 3.1. *In the category $\mathcal{O}_\lambda$ for any standard object $\Delta(\nu)$ and any costandard object $\nabla(\mu)$ we have $\operatorname{Ext}^n(\Delta(\nu), \nabla(\mu)) = 0$ for all $n > 0$.*

Proof. For $n = 1$ that was proved in Daniil's talk. For $n > 1$ we will prove it by the decreasing induction on ν. Suppose that we have proved it for all $\nu' > \nu$. The projective cover $P(\nu)$ of $\Delta(\nu)$ has a standard filtration such that $K := \operatorname{Ker}(P(\nu) \to \Delta(\nu))$ is filtered by $\Delta(\nu')$ for $\nu' > \nu$. Applying long exact sequences at every step of the filtration we get $\operatorname{Ext}^n(K, \nabla(\mu)) = 0$. Now for the short exact sequence $0 \to K \to P(\nu) \to \Delta(\nu) \to 0$ we have the corresponding long exact sequence $0 = \operatorname{Ext}^{n-1}(K, \nabla(\mu)) \to \operatorname{Ext}^n(\Delta(\nu), \nabla(\mu)) \to \operatorname{Ext}^n(P(\nu), \nabla(\mu)) = 0$. That implies the proposition. $\square$

We will give a criterion for an object $M \in \mathcal{O}$ to be standardly filtered.

Proposition 3.2. *An object $M \in \mathcal{O}$ is standardly filtered iff $\operatorname{Ext}^1(M, \nabla_j) = 0$ for all j.*

Proof. "$\Rightarrow$": This implication is an exercise.

"$\Leftarrow$": We will use the induction on the number of simple objects in M. Let λ be a maximal weight such that $L(\lambda)$ is a composition factor in M. We set $\mathcal{C}$ to be the subcategory spanned by all simples $L(\mu)$ for $\mu \not\geq \lambda$. Let N be the maximal quotient of M that lies in $\mathcal{C}$ and let K be the kernel $0 \to K \to M \to N \to 0$. Note that K has no nonzero quotients lying in $\mathcal{C}$. For any μ we have the following long exact sequence $\operatorname{Hom}(K, \nabla(\mu)) \to \operatorname{Ext}^1(N, \nabla(\mu)) \to \operatorname{Ext}^1(M, \nabla(\mu))$. If $\mu \not\geq \lambda$, then $\operatorname{Hom}(K, \nabla(\mu)) = \operatorname{Ext}^1(M, \nabla(\mu)) = 0$, so $\operatorname{Ext}^1(N, \nabla(\mu)) = 0$. By the induction hypothesis, N is standardly filtered. Therefore $\operatorname{Ext}^2(N, \nabla(\mu)) = 0$. We have an exact sequence $\operatorname{Ext}^1(M, \nabla(\mu)) \to \operatorname{Ext}^1(K, \nabla(\mu)) \to \operatorname{Ext}^2(N, \nabla(\mu))$ that implies $\operatorname{Ext}^1(K, \nabla(\mu)) = 0$. Let $L(\lambda)^k$ be the maximal semi-simple quotient of K. The surjective map $K \to L(\lambda)^k$ induces a map $\Delta(\lambda)^k \to K$ because $\Delta(\lambda)$ is projective in the category spanned by $L(\lambda)$ and $\mathcal{C}$. The cokernel of this map is in $\mathcal{C}$, therefore it is 0. Let K_1 be a kernel of this map, so $0 \to K_1 \to \Delta(\lambda)^k \to K \to 0$ is an exact sequence. Note that $K_1 \in \mathcal{C}$. For any $\mu \not\geq \lambda$ we have the following exact sequence $0 = \operatorname{Hom}(\Delta(\lambda)^k, \nabla(\mu)) \to \operatorname{Hom}(K_1, \nabla(\mu)) \to \operatorname{Ext}^1(K, \nabla(\mu)) = 0$, so $\operatorname{Hom}(K_1, \nabla(\mu)) = 0$. But then $K_1 = 0$, so $\Delta(\lambda)^k \to K$ is an isomorphism. Therefore M has a standard filtration as an extension of N by K. $\square$

3.2. **Proof ot Theorem 2.3.** Let us give a plan of the proof first. Recall that $\mathbb{V}^*$ stands for the right adjoint of $\mathbb{V}$. We set $T := T_{0\to-\rho}$ and $T^* := T_{-\rho\to 0}$. We need to prove that the natural map $\mathrm{Hom}(M,N) \to \mathrm{Hom}(\mathbb{V}(M), \mathbb{V}(N)$ is an isomorphism when M, N are projective. As this map factors through $\mathrm{Hom}(M,N) \to \mathrm{Hom}(M, \mathbb{V}^*\mathbb{V}(N)) \simeq \mathrm{Hom}(\mathbb{V}(M), \mathbb{V}(N)$ it is suffices to prove that $\mathbb{V}^*\mathbb{V}(M) \simeq M$ for any projective M. From Proposition 2.2 $P_{min} \simeq \mathbb{V}^*\mathbb{V}(P_{min})$. We will show that any projective module M is isomorphic to the kernel of a map $P_{min}^{\oplus k} \to P_{min}^{\oplus n}$. Applying the left exact functor $\mathbb{V}^*\mathbb{V}$ we present of $\mathbb{V}^*\mathbb{V}(M)$ as the kernel of the same map, so $\mathbb{V}^*\mathbb{V}(M) \simeq M$.

Let us start with the claim that any projective module M can be presented as the kernel of a map $P_{min}^{\oplus k} \to P_{min}^{\oplus n}$. We state that the injective map into $P_{min}^{\oplus k}$ can be given by applying the adjunction unit map $M \to T^*T(M)$.

Lemma 3.3. *For a standardly filtered object M the adjunction unit map $M \to T^*T(M)$ is injective. Analogously for a costandardly filtered object N the adjunction counit map $TT^*(N) \to N$ is surjective.*

Proof. We will prove the injectivity of the adjunction unit. The second statement is just dual.

We know that the map $T(M) \to TT^*T(M)$ is an injection because T^* is right adjoint to T. Therefore the kernel of the adjunction unit is anihilated by T. The socle of any standardly filtered module M is a direct sum of some copies of $\Delta(w_0 \cdot 0)$. But $T(\Delta(w \cdot 0)) = \Delta(-\rho)$, so for any N in a socle of M we have $T(N) \neq 0$. Therefore the adjunction unit $M \to T^*T(M)$ is an injection. □

Corollary 3.4. *For any standardly filtered M we have $T^*T(M) \simeq P_{min}^{\oplus k}$ for some k, so M can be embedded in $P_{min}^{\oplus k}$.*

Lemma 3.5. *For any projective object $P \in \mathcal{O}_0$, the quotient $T^*T(P)/P$ has a standard filtration.*

Proof. By Proposition 3.2, $T^*T(P)/P$ has a standard filtration iff $\mathrm{Ext}^1(T^*T(P)/P, \nabla(w \cdot 0))$ for all $w \in W$. For the short exact sequence $0 \to P \to T^*T(P) \to T^*T(P)/P \to 0$ we consider the corresponding long exact sequence.

$$\mathrm{Hom}(T^*T(P), \nabla(w\cdot 0)) \to \mathrm{Hom}(P, \nabla(w\cdot 0)) \to \mathrm{Ext}^1(T^*T(P)/P, \nabla(w\cdot 0)) \to \mathrm{Ext}^1(T^*T(P), \nabla(w\cdot 0)).$$

The object $T^*T(P)$ is projective, so $\mathrm{Ext}^1(T^*T(P), \nabla(w \cdot 0)) = 0$.

Therefore it is enough to show that the map $\mathrm{Hom}(T^*T(P), \nabla(w \cdot 0)) \to \mathrm{Hom}(P, \nabla(w \cdot 0))$ is surjective. By the biadjointness $\mathrm{Hom}(T^*T(P), \nabla(w \cdot 0)) \simeq \mathrm{Hom}(P, T^*T(\nabla(w \cdot 0)))$. Now the map $\mathrm{Hom}(P, T^*T(\nabla(w \cdot 0))) \to \mathrm{Hom}(P, \nabla(w \cdot 0))$ is induced by the adjunction counit $T^*T(\nabla(w \cdot 0)) \to \nabla(w \cdot 0)$. This map is surjective by Lemma 3.3.

Since P is projective $\mathrm{Hom}(P, T^*T(\nabla(w\cdot 0))) \to \mathrm{Hom}(P, \nabla(w\cdot 0))$ is surjective, so $\mathrm{Hom}(T^*T(P), \nabla(w\cdot 0)) \to \mathrm{Hom}(P, \nabla(w \cdot 0))$ is a surjection. Therefore $\mathrm{Ext}^1(T^*T(P)/P, \nabla(w \cdot 0)) = 0$ and the lemma follows. □

Corollary 3.6. *For any projective object $P \in \mathcal{O}_0$ there is an exact sequence $0 \to P \to P_{min}^{\oplus k} \to P_{min}^{\oplus n}$ for some k and n.*

Proof. By Corollary 3.4 we can embed P into $T^*T(P) \simeq P_{min}^k$ for some k. The cokernel of this map by Lemma 3.5 is standardly filtered. Applying Corollary 3.4 for P_{min}^k/P we finish the proof. □

Proposition 3.7. *For any projective module M we have $\mathbb{V}^*\mathbb{V}(M) \simeq M$.*

Proof. Let M be the kernel of a map $\varphi : P_{min}^{\oplus k} \to P_{min}^{\oplus n}$. Under the identification $P_{min} \xrightarrow{\sim} \mathbb{V}^*\mathbb{V}(P_{min})$ we have $\mathbb{V}^*\mathbb{V}(\varphi) = \varphi$. Since $\mathbb{V}^*$ is right adjoint, it is left exact. Hence the functor $\mathbb{V}^*\mathbb{V}$ is left exact, so $\mathbb{V}^*\mathbb{V}(M)$ is the kernel of φ.

□

Corollary 3.8. *The functor $\mathbb{V}$ is fully faithful on projectves.*

4. Soergel's functor vs reflection functor

In this section we prove that $\mathbb{V} : \mathcal{O}_0 \to C$-mod intertwines the reflection functor $\mathcal{P}_i$ with $\bullet \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} \mathbb{C}[\mathfrak{h}]$.

The scheme of the proof is as follows.

1) Let μ be an integral singular element of $\mathfrak{h}^*$ such that $\mu+\rho$ is dominant. So we have the extended translation functors $\tilde{T}_{0\to\mu} : \mathcal{O}_0 \to \tilde{\mathcal{O}}_\mu$ and $\tilde{\mathbb{V}}_\mu := \tilde{T}_{\mu\to-\rho} : \tilde{\mathcal{O}}_\mu \to C$-mod. By transitivity, $\mathbb{V} = \tilde{\mathbb{V}}_\mu \circ \tilde{T}_{0\to\mu}$. Let $\tilde{P}_{min,\mu}$ denote the projective cover of $L_{min,\mu}$ in $\tilde{\mathcal{O}}_\mu$. We will show that the functors $\tilde{T}_{0\to\mu}, \mathbb{V}_\mu$ induce isomorphisms $\mathrm{End}_{\mathcal{O}}(P_{min}) \xrightarrow{\sim} \mathrm{End}_{\tilde{\mathcal{O}}}(\tilde{P}_{min,\mu}) \xrightarrow{\sim} C$.

2) Let $P_{min,\mu}$ denote the projective cover of L_{min} in $\mathcal{O}_\mu$ so that $\tilde{P}_{min,\mu} = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{W_\mu}} P_{min,\mu}$. We will use this description of $\tilde{P}_{min,\mu}$ together with 1) to identify $\mathrm{End}_{\mathcal{O}}(P_{min,\mu})$ with C^{W_μ} so that we get the functor $\mathbb{V}_\mu = \mathrm{Hom}_{\mathcal{O}}(P_{min,\mu}, \bullet) : \mathcal{O}_\mu \to C^{W_\mu}$-mod.

3) We will deduce from 1) and 2) that the following diagram is commutative.

$$\begin{tikzcd} \mathcal{O}_0 \arrow[r, "T_{0\to\mu}"] \arrow[d, "\mathbb{V}"'] & \mathcal{O}_\mu \arrow[d, "\mathbb{V}_\mu"] \\ C\text{-mod} \arrow[r, "\mathsf{frg}"] & C^{W_\mu}\text{-mod} \end{tikzcd} .$$

4) We use 3) together with the adjointness and the second Soergel theorem to show that the following diagram is commutative.

$$\begin{tikzcd} \mathcal{O}_\mu \arrow[r, "T_{\mu\to 0}"] \arrow[d, "\mathbb{V}_\mu"'] & \mathcal{O}_0 \arrow[d, "\mathbb{V}"] \\ C^{W_\mu}\text{-mod} \arrow[r, "\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^{W_\mu}}\bullet"] & C\text{-mod} \end{tikzcd} .$$

Then the proof of Theorem 2.4 will follow from steps 3 and 4 and the definition of $\mathcal{P}_i$ as $T_{\mu\to 0} \circ T_{0\to\mu}$, where i is the only index with $\langle \mu+\rho, \alpha_i^\vee\rangle$.

4.1. Endomorphisms of $\tilde{P}_{min,\mu}$.

Steps 1 and 4 of the proof will need the following lemma.

Lemma 4.1. $T_{0\to\mu}(P_{min}) = P_{min,\mu}^{|W_\mu|}$.

Proof. These two objects have the same K_0-classes: both are equal to $\sum_{w\in W}[\Delta(w\cdot\mu)]$. But the classes of indecomposable projectives in K_0 are linearly independent because of the upper triangularity property for projectives. So if the classes of two projectives are equal, then the projectives are isomorphic. $\square$

Proposition 4.2. *The following is true:*

(1) *The functor $\tilde{T}_{0\to\mu} : \mathcal{O}_0 \to \tilde{\mathcal{O}}_\mu$ maps P_{min} to $\tilde{P}_{min,\mu}$ and induces an isomorphism $\mathrm{End}(P_{min}) \xrightarrow{\sim} \mathrm{End}(\tilde{P}_{min,\mu})$.*

(2) *Similarly, the functor $\tilde{\mathbb{V}}_\mu : \tilde{\mathcal{O}}_\mu \to C$-mod maps $\tilde{P}_{min,\mu}$ to C and induces an isomorphism $\mathrm{End}_{\tilde{\mathcal{O}}}(\tilde{P}_{min,\mu}) \xrightarrow{\sim} C$.*

Proof. Let us prove (1).

Lemma 4.3. *Let w_0^μ be the longest element of W^μ. We set $u = w_0 w_0^\mu$, so that $\Delta(u\cdot 0)$ is the bottom factor in the standard filtration of $T_{\mu\to 0}(\Delta_{min,\mu})$. Then we have $\tilde{T}^*_{0\to\mu}(\Delta_{min,\mu}) = \Delta(u\cdot 0)$.*

Proof. Analogously to Proposition 3.18 of the last talk, $\Delta(u\cdot 0)$ is the intersection of kernels of all α_i acting on $T_{\mu\to 0}(\Delta_{min,\mu})$ for i such that $\langle\mu+\rho,\alpha_i^\vee\rangle = 0$. For every such i we have a filtration of $T_{\mu\to 0}(\Delta_{min,\mu})$ by $\tilde{\Delta}_{w,i} = T_{\mu_i\to 0}(\Delta(w\cdot\mu_i))$ where $\langle\mu_i+\rho,\alpha_j^\vee\rangle = 0$ iff $i=j$. On each of $\tilde{\Delta}_{w,i}$

the action of α_i kills only the bottom factor. Therefore the intersection of kernels of all α_i is the bottom factor. $\square$

Applying this lemma we get
$\dim \operatorname{Hom}(\tilde{T}_{0\to\mu}(P_{min}), \Delta_{min,\mu}) = \dim \operatorname{Hom}(P_{min}, \tilde{T}^*_{0\to\mu}(\Delta_{min,\mu})) = \dim \operatorname{Hom}(P_{min}, \Delta(u\cdot 0)) = 1$ where the last equality holds by Corollary 3.6 of the previous talk. Analogously to the final part of the proof of Theorem 2.1 we have a surjective map $\phi : \tilde{P}_{min,\mu} \to \tilde{T}_{0\to\mu}(P_{min})$. Therefore it is enough to show that the induced map on restrictions to $\mathcal{O}_\mu$ is an isomorphism. But that follows from Lemma 4.1. The isomorphism $\tilde{T}_{0\to\mu} : P_{min} \xrightarrow{\sim} \tilde{P}_{min,\mu}$ yields $\operatorname{End}_{\mathcal{O}}(P_{min}) \xrightarrow{\sim} \operatorname{End}_{\tilde{\mathcal{O}}}(\tilde{P}_{min,\mu})$.

Let us prove (2). We know that $\tilde{\mathbb{V}}_\mu \circ \tilde{T}_{0\to\mu}(P_{min}) = C$ from the previous lecture, and also that $\tilde{T}_{0\to\mu}(P_{min}) = \tilde{P}_{min,\mu}$. It follows that $\tilde{\mathbb{V}}_\mu(\tilde{P}_{min,\mu}) = C$. And also we know that $\mathbb{V} = \tilde{\mathbb{V}}_\mu \circ \tilde{T}_{0\to\mu}$ gives rise to an isomorphism $\operatorname{End}_{\mathcal{O}}(P_{min}) \xrightarrow{\sim} C$. Together with (1) this implies that $\tilde{\mathbb{V}}_\mu$ gives rise to an isomorphism $\operatorname{End}(\tilde{P}_{min,\mu}) \xrightarrow{\sim} C$. $\square$

4.2. **Endomorphisms of $P_{min,\mu}$.**

Lemma 4.4. *We have a natural isomorphism* $\operatorname{End}_{\mathcal{O}}(P_{min,\mu}) \cong C^{W_\mu}$.

Proof. Recall that we can view $\tilde{P}_{min,\mu}$ as an object in $\mathcal{O}_\mu$ (formally, via the forgetful functor Res_μ), the group W_μ acts on $\tilde{P}_{min,\mu} \in \mathcal{O}_\mu$ by automorphisms, and $P_{min,\mu} = \tilde{P}^{W_\mu}_{min,\mu}$. We have

$$C = \operatorname{Hom}_{\tilde{\mathcal{O}}_\mu}(\tilde{P}_{min,\mu}, \tilde{P}_{min,\mu}) = \operatorname{Hom}_{\tilde{\mathcal{O}}_\mu}(\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{W_\mu}} P_{min,\mu}, \tilde{P}_{min,\mu}) =$$
$$= \operatorname{Hom}_{\mathcal{O}_\mu}(P_{min,\mu}, \tilde{P}_{min,\mu}).$$

By the first paragraph of the proof, W_μ acts on the final expression and the invariants are $\operatorname{Hom}_{\mathcal{O}_\mu}(P_{min,\mu}, P_{min,\mu})$. Both actions of W_μ on the right hand side and on C corresponds to the diagonal action on $\operatorname{Hom}_{\tilde{\mathcal{O}}_\mu}(\tilde{P}_{min,\mu}, \tilde{P}_{min,\mu})$. The lemma follows. $\square$

4.3. **The functor $\mathbb{V}$ vs projection to the wall.** Let $\mathsf{frg}_\mu : C\text{-mod} \to C^{W_\mu}\text{-mod}$ denote the forgetful functor. First we note that

$$\mathsf{frg}_\mu \circ \tilde{\mathbb{V}}_\mu \cong \mathbb{V}_\mu \circ \operatorname{Res}_\mu : \tilde{\mathcal{O}}_\mu \to C^{W_\mu}\text{-mod}. \tag{1}$$

Indeed, by the construction of the isomorphism $\operatorname{End}_{\mathcal{O}}(P_{min,\mu}) \cong \operatorname{End}_{\tilde{\mathcal{O}}}(\tilde{P}_{min,\mu})^{W_\mu}$ in the previous subsection, both functors in (1) are $\operatorname{Hom}_{\mathcal{O}}(P_{min,\mu}, \operatorname{Res}_\mu(\bullet))$.

From here we deduce that $\mathbb{V}_\mu \circ T_{0\to\mu} \cong \mathsf{frg}_\mu \circ \mathbb{V}$ from (1). Indeed, $T_{0\to\mu} = \operatorname{Res}_\mu \circ \tilde{T}_{0\to\mu}$. So

$$\mathbb{V}_\mu \circ T_{0\to\mu} = \mathbb{V}_\mu \circ \operatorname{Res}_\mu \circ \tilde{T}_{0\to\mu} = [(1)] = \mathsf{frg}_\mu \circ \tilde{V}_\mu \circ \tilde{T}_{0\to\mu} = \mathsf{frg}_\mu \circ \mathbb{V}.$$

4.4. **The functor $\mathbb{V}$ vs translation from the wall.** In this subsection we will deduce

$$\mathbb{V} \circ T_{\mu\to 0}(\bullet) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{W_\mu}} \mathbb{V}_\mu(\bullet), \tag{2}$$

an equality of functors $\mathcal{O}_\mu \to C\text{-mod}$, from $\mathbb{V}_\mu \circ T_{0\to\mu} \cong \mathsf{frg}_\mu \circ \mathbb{V}$. Let us take the left adjoint of the previous equality, we get $T_{\mu\to 0} \circ \mathbb{V}^!_\mu \cong \mathbb{V}^! \circ \mathsf{frg}^!_\mu$. Now compose with $\mathbb{V}$ on the left. As for any finitely generated C-module M there is an exact sequence $C^n \to C^k \to M \to 0$ we can analogously to Proposition 3.7 show that the adjunction unit map $M \to \mathbb{V} \circ \mathbb{V}^!(M)$ is an isomorphism. So we get $\mathbb{V} \circ T_{\mu\to 0} \circ \mathbb{V}^!_\mu \cong \mathsf{frg}^!_\mu$. Now compose with $\mathbb{V}_\mu$ on the right to get $\mathbb{V} \circ T_{\mu\to 0} \circ \mathbb{V}^!_\mu \circ \mathbb{V}_\mu = \mathsf{frg}^!_\mu \circ \mathbb{V}_\mu$. In the right hand side we already have the right hand side of (2) because $\mathsf{frg}^!_\mu = C \otimes_{C^{W_\mu}} \bullet \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{W_\mu}} \bullet$.

So to establish (2), we need to show that $\mathbb{V} \circ T_{\mu\to 0} \circ \mathbb{V}^!_\mu \circ \mathbb{V}_\mu \cong \mathbb{V} \circ T_{\mu\to 0}$. We have the adjunction counit morphism $\mathbb{V}^!_\mu \circ \mathbb{V}_\mu \to \operatorname{id}$. This gives rise to a functor morphism $\mathbb{V} \circ T_{\mu\to 0} \circ \mathbb{V}^!_\mu \circ \mathbb{V}_\mu \to \mathbb{V} \circ T_{\mu\to 0}$. We need to show that $\mathbb{V} \circ T_{\mu\to 0}$ annihilates the kernel and the cokernel of $\mathbb{V}^!_\mu \circ \mathbb{V}_\mu(M) \to M$. The induced morphism $\mathbb{V}_\mu \circ \mathbb{V}^!_\mu \circ \mathbb{V}_\mu(M) \to \mathbb{V}_\mu(M)$ is an isomorphism (it is surjective by the standard properties of adjointness and it is injective because $\mathrm{id} \xrightarrow{\sim} \mathbb{V}_\mu \circ \mathbb{V}^!_\mu$, which is proved in the same way as for the functor $\mathbb{V}$). So the kernel and the cokernel of $\mathbb{V}^!_\mu \circ \mathbb{V}_\mu(M) \to M$ are killed by $\mathbb{V}_\nu$.

We claim that if $\mathrm{Hom}_\mathcal{O}(P_{min,\mu}, L) = 0$, then $\mathrm{Hom}_\mathcal{O}(P_{min}, T_{\mu\to 0}(L)) = 0$, this will finish the proof of (2). Note that $\mathrm{Hom}_\mathcal{O}(T_{0\to\mu}(P_{min}), L) = \mathrm{Hom}_\mathcal{O}(P_{min}, T_{\mu\to 0}(L))$. By Lemma 4.1 $T_{0\to\mu}P_{min} \cong P^{|W_\mu|}_{min,\mu}$. The claim follows.

4.5. **Proof of Theorem 2.4.** Now we know that $\mathbb{V}_\mu \circ T_{0\to\mu} \cong \mathsf{frg}_\mu \circ \mathbb{V}$ and $\mathbb{V}\circ T_{\mu\to 0} \cong \mathbb{V}(\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{W_\mu}} \bullet)$. Take μ such that i is the only index with $\langle \mu + \rho, \alpha_i^\vee\rangle = 0$. Then

$$\begin{aligned}\mathbb{V} \circ \mathcal{P}_i(\bullet) &\cong \mathbb{V} \circ T_{\mu\to 0} \circ T_{0\to\mu}(\bullet) \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} (\mathbb{V}_\mu \circ T_{0\to\mu}(\bullet)) \\ &\cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_i}} \mathbb{V}(\bullet).\end{aligned}$$

5. Soergel modules and bimodules

We are interested in the image of projectives under the Soergel's functor $\mathbb{V}$.

Definition 5.1. *For a sequence $\underline{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_k})$ of simple reflections we set $BS_{\underline{w}} := \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_{i_1}}} \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^{s_{i_2}}} \dots \otimes_{\mathbb{C}[\mathfrak{h}]^{s_{i_k}}} \mathbb{C}[\mathfrak{h}]$. This is called a Bott-Samelson bimodule. By a Bott-Samelson module we mean $BS_{\underline{w}} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}$ for some $\underline{w}$.*

Definition 5.2. *We define the category* SBim *of Soergel bimodules as the minimal subcategory in the category of graded $\mathbb{C}[\mathfrak{h}]$-bimodules closed under taking direct sums, direct graded summands and shifts of grading containing all Bott-Samelson bimodules. Morphisms in* SBim *are graded morphisms of $\mathbb{C}[\mathfrak{h}]$-bimodules.*

Similarly, we define the category SMod *of Soergel modules as the subcategory in the category of graded left $\mathbb{C}[\mathfrak{h}]$-modules closed under taking direct sums, direct graded summands and shifts of grading containing all Bott-Samelson modules. Morphisms in* SBim *are graded morphisms of left $\mathbb{C}[\mathfrak{h}]$-modules.*

Definition 5.3. *We define the category* SMod_{ungr} *of ungraded Soergel modules as the category with the same set of objects as in* SMod *and $\mathbb{C}[\mathfrak{h}]$-linear morphisms that do not necessarily preserve grading.*

Theorem 5.4. *The functor $\mathbb{V}$ gives an equivalence of the subcategory $\mathcal{O}_0 - proj \subset \mathcal{O}_0$ consisting of projective objects in the principal block of category $\mathcal{O}$ and* SMod_{ungr}.

To prove this theorem we need to compare indecomposable objects of SMod and SMod_{ungr}. We will use the following proposition.

Proposition 5.5. *Let A be a positively graded finite dimensional algebra over $\mathbb{C}$ and let M be a graded finite dimensional A-module. If M is indecomposable as a graded module, then it's indecomposable as a module.*

Proof. Consider the algebra $\mathrm{End}_A(M)$ of all A-linear endomorphisms of M, it is finite dimensional and graded. The radical R is graded. Indeed, the grading gives an action of the one dimensional torus $\mathbb{C}^\times$ on $\mathrm{End}_A(M)$ by automorphisms. Then the quotient $\mathrm{End}_A(M)/\phi_t(R) = \phi_t(\mathrm{End}_A(M)/R)$ is semi-simple. Therefore $R \subset \phi_t(R)$ and so radical is graded.

The module M is indecomposable iff $M/R = \mathbb{C}$. Indeed, if $M = M_1 \oplus M_2$ then $x\,\mathrm{id}_{M_1} + y\,\mathrm{id}_{M_2} \notin R$ for any $(x, y) \neq (0, 0)$. This quotient Q is a semi-simple algebra, so it is isomorphic to a direct sum of matrix algebras. As R is graded the quotient Q is equipped with an algebra grading.

Lemma 5.6. *Let F be an algebraically closed field of characteristic 0. Let B be the direct sum of matrix algebras over F equipped with an algebra grading. Then the degree 0 part is the sum of matrix algebras of the same total rank as B.*

Proof. Every grading of the matrix algebra over a characteristic 0 algebraically closed field is inner because any derivation is inner. So the same holds for a direct sum of matrix algebras as well. If $\mathrm{ad}(x)$ defines a grading then x is diagonalizable and up to adding a central element has integral eigenvalues. Elements of degree 0 are exactly elements commuting with x. The centralizer of x in a matrix algebra is the direct sum of the endomorphism algebras of the eigenspaces. The lemma follows. □

If M is decomposable then $\dim Q \geq 2$ and the lemma implies that we have a degree 0 non-trivial idempotent in Q. Now we can lift it to an idempotent in the degree 0 part of $\mathrm{End}_A(M)$. But there are no such nontrivial idempotents since M is indecomposable as a graded module.

□

Applying this proposition to $A = C$ we get the following corollary.

Corollary 5.7. *Indecomposable summands of Bott-Samelson modules are isomorphic to its indecomposable graded summands.*

5.1. **Proof of Theorem 5.4.** Let $\underline{w} = s_{i_1} \dots s_{i_k}$ be a reduced expression of $w \in W$. We set $P_{\underline{w}} := \mathcal{P}_{i_k} \circ \mathcal{P}_{i_{k-1}} \circ \dots \circ \mathcal{P}_{i_1}(\Delta(0))$. By Theorem 6.3 from Chris's talk, $P_{\underline{w}} = P(w \cdot 0) \oplus \bigoplus P(w' \cdot 0)$ for some $w' \prec w$. By Theorem 2.4 $BS_{\underline{w}} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C} = \mathbb{V}(P_{\underline{w}})$. Therefore $BS_{\underline{w}} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C} = \mathbb{V}(P(w \cdot 0)) \oplus \bigoplus \mathbb{V}(P(w' \cdot 0))$. We claim that every summand $\mathbb{V}(P(w' \cdot 0))$ is indecomposable. Indeed, by Theorem 2.3 $\mathrm{End}_C(\mathbb{V}(P(w' \cdot 0))) \simeq \mathrm{End}_{\mathcal{O}}(P(w' \cdot 0))$. But the latter endomorphism algebra does not have nontrivial idempotents because $P(w' \cdot 0)$ is indecomposable.
Then I have a decomposition of $BS_{\underline{w}}$ in the direct sum of indecomposables $\mathbb{V}(P(w' \cdot 0))$. On the other hand I have a decomposition of $BS_{\underline{w}}$ in the direct sum of indecomposable (as graded modules) Soergel modules. By Corollary 5.7 the latter one is also a decomposition in the direct sum of indecomposables. But by the Krull-Schmidt theorem for modules there is a unique decomposition of $\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} BS_{\underline{w}}$ in the direct sum of indecomposables up to a permutation. Therefore $\mathbb{V}(\mathcal{O}_0 - proj) = \mathrm{SMod}_{ungr}$. In other words, $\mathbb{V} : \mathcal{O}_0 - proj \to \mathrm{SMod}_{ungr}$ is essentially surjective on objects. By Theorem 2.3 $\mathbb{V}$ is fully faithful. So it is a category equivalence.

Corollary 5.8. *Indecomposable objects S_w in SMod_{ungr} are labelled by elements of W. We have $\mathbb{V}(P(w \cdot 0)) = S_w$.*

Corollary 5.9. *We have $\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} BS_{\underline{w}} = S_w \oplus \bigoplus S_{w'}$ for some $w' \prec w$.*

Corollary 5.10. *Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$ be a Lie algebra of type B_n and $\mathfrak{g}' = \mathfrak{sp}_{2n}$ a Lie algebra of type C_n. Then principal blocks $\mathcal{O}_0$ and $\mathcal{O}_0'$ of corresponding categories $\mathcal{O}$ are equivalent.*

Proof. By Theorem 5.4 we have equivalences $\mathcal{O}_0 - proj \simeq \mathrm{SMod}_{ungr,\mathfrak{g}}$ and $\mathcal{O}_0' - proj \simeq \mathrm{SMod}_{ungr,\mathfrak{g}'}$. But categories $\mathrm{SMod}_{ungr,\mathfrak{g}}$ and $\mathrm{SMod}_{ungr,\mathfrak{g}'}$ depend only on the Weyl group W and therefore coincide. Therefore $\mathcal{O}_0 - proj \simeq \mathcal{O}_0' - proj$. The corollary follows. □

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