LECTURE 4: ELLIPTIC HALL ALGEBRA BY GENERATORS AND RELATIONS

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Abstract. These are notes for a talk given at the MIT-Northeastern Graduate Student Seminar on Double Affine Hecke Algebras and Elliptic Hall Algebras, Spring 2017.

Contents

1. Goals and structure of the talk

The main goal of the talk is to introduce the elliptic Hall algebra (EHA) and show that it is isomorphic to a quotient of the Ding-Iohara algebra (also known as the quantum toroidal $\mathfrak{gl}_1$). We will start with the notion of quantum affinization for Kac-Moody algebras. After that we will define the elliptic Hall algebra $\mathcal{E}_K$ and its specialization $\mathcal{E}$ that will be one of the main objects of study in our seminar. The first one can be understood as the quantum affinization of $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$. Next we will move to the Ding-Iohara algebra $\tilde{\mathcal{U}}$ and its quotient $\tilde{\mathcal{E}}$ that is the quantum toroidal $\mathfrak{gl}_1$. We will define it by generators and relations and construct a surjective map $\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{E}$. The main result of this talk is that this map is an isomorphism. We will give a combinatorial proof of this theorem. Interaction with other objects such as the Hall algebra of coherent sheaves on an elliptic curve and shuffle algebra will be discussed in other talks. We finish these notes with a discussion of Hopf algebra structure on the EHA.

2. Quantum affinization

2.1. Quantum Kac-Moody algebra. Let $\mathfrak{g}$ be a Kac-Moody algebra. We set C be the Cartan matrix of $\mathfrak{g}$, $\mathfrak{h}$ – its coroot lattice. Let $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ and $\alpha_1^\vee, \ldots, \alpha_n^\vee \in \mathfrak{h}$ be sets of simple roots and simple coroots correspondingly. For this data we define the quantum Kac-Moody algebra $\mathcal{U}_q(\mathfrak{g})$.

Definition 2.1.1. *The quantum Kac-Moody algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra generated by elements k_h for every $h \in \mathfrak{h}$ and $x_1^\pm, x_2^\pm, \ldots, x_n^\pm$ with the following set of relations.*

$$\begin{aligned}
&k_{h+h'} = k_h k_{h'},\ k_0 = 1,\\
&k_h x_i^\pm = q^{\pm\alpha_i(h)} x_i^\pm k_h,\\
&[x_i^+, x_j^-] = \delta_{i,j} \frac{k_{\alpha_i^\vee} - k_{-\alpha_i^\vee}}{q - q^{-1}},\\
&\sum_{r=0,1,\ldots,1-C_{i,j}} \frac{(-1)^r}{[r!][(1-C_{i,j}-r)!]} \left(x_i^\pm\right)^{1-C_{i,j}-r} x_j^\pm \left(x_i^\pm\right)^r = 0,\ \textit{for } i \neq j.
\end{aligned}$$

In the formulas above we put $[m] = \frac{q^m - q^{-m}}{q-q^{-1}}$ and $[m]! = [m][m-1]\ldots[1]$.

2.1.1. *Quantum Kac-Moody algebra of $\mathfrak{sl}_2$.* Let us show an example and apply the construction above to the Lie algebra $\mathfrak{sl}_2$. We have elements $x_1^+ = e$, $x_1^- = f$ and k_m for $m \in \mathbb{Z}$. Note that $k_m = k_1^m$. Then $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by $K := k_1$, K^{-1}, e and f with the following set of relations.

$$\begin{aligned}
&KK^{-1} = K^{-1}K = 1,\\
&Ke = q^2 eK,\\
&Kf = q^{-2} fK,\\
&[e, f] = \frac{K - K^{-1}}{q - q^{-1}}.
\end{aligned}$$

2.2. Quantum affinization. Let $\mathfrak{g}$, $\mathfrak{h}$, α_i and $\alpha_i^\vee$ be the same as in the previous section. Let us have variables $x_{i,r}^\pm$ (with $i \in \{1, \ldots, n\}$, $r \in \mathbb{Z}$), $h_{i,m}$ (with $i \in \{1, \ldots, n\}$, $m \in \mathbb{Z} \setminus \{0\}$). We put $\phi_{i,m}^\pm$ be the elements determined by the formal power series

$$\begin{aligned}
&\sum_{m \geq 0} \phi_{i,\pm m}^\pm z^{\pm m} = k_{\alpha_i^\vee} \exp\left(\pm(q - q^{-1}) \sum_{m' \geq 1} h_{i,\pm m'} z^{\pm m'}\right),\\
&\phi_{i,m}^+ = 0,\ \text{for } m < 0,\\
&\phi_{i,m}^- = 0,\ \text{for } m > 0.
\end{aligned}$$

Let us consider the following series:

$$\begin{aligned}
&x_i^\pm(w) = \sum_{r \in \mathbb{Z}} x_{i,r}^\pm w^r,\\
&\phi_i^\pm(z) = \sum_{m \in \mathbb{Z}_{>0}} h_{i,m}^\pm z^m,\\
&\delta\left(\frac{z}{w}\right) = \sum_{k \in \mathbb{Z}} \left(\frac{z}{w}\right)^k.
\end{aligned}$$

Definition 2.2.1. *The quantum affinization of $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ with the generators $x^{\pm}_{i,r}$, k_h (with $h \in \mathfrak{h}$), $h_{i,m}$ and the relations below.*

$$
\begin{aligned}
&(i) k_{h+h'} = k_h k_{h'},\ k_0 = 1,\ k_h \phi^{\pm}_i(z) = \phi^{\pm}_i(z) k_h,\\
&(ii) \phi^{\pm}_i(w)\phi^{\pm}_j(z) = \phi^{\pm}_j(z)\phi^{\pm}_i(w),\ \phi^{+}_i(w)\phi^{-}_j(z) = \phi^{-}_j(z)\phi^{+}_i(w),\\
&(iii) k_h x^{\pm}_i(z) = q^{\pm\alpha_i(h)} x^{\pm}_i(z) k_h,\\
&(iv) \phi^{\pm}_i(z) x^{\pm}_j(w) = \frac{q^{\pm C_{i,j}} w - z}{w - q^{\pm C_{i,j}} z} x^{\pm}_j(w)\phi^{\pm}_i(z),\\
&(v) [x^{+}_i(z), x^{-}_k(w)] = \frac{\delta_{i,j}}{q - q^{-1}} \delta(\frac{w}{z}) \left(\phi^{+}_i(w) - \phi^{-}_i(w)\right),\\
&(vi) (w - q^{\pm C_{i,j}} z) x^{\pm}_i(z) x^{\pm}_j(w) = (q^{\pm C_{i,j}} w - z) x^{\pm}_j(w) x^{\pm}_i(z),\\
&(vii) \sum_{\sigma \in \Sigma_s} \sum_{k=0,\dots,s} (-1)^k \frac{[s]!}{[k]![s-k]!} x^{\pm}_i(w_{\sigma(1)}) \dots x^{\pm}_i(w_{\sigma(k)}) x^{\pm}_i(z) x^{\pm}_i(w_{\sigma(k+1)}) \dots x^{\pm}_i(w_{\sigma(s)}) = 0,
\end{aligned}
$$

where $s = 1 - C_{i,j}$. The equation (iv) is expanded for $|w| > |z|$.

Remark 2.2.2. *The correspondence $x^{\pm}_i \to x^{\pm}_{i,0}$ gives a map of algebras $\mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\hat{\mathfrak{g}})$. Therefore $\mathcal{U}_q(\hat{\mathfrak{g}})$ has a structure of $\mathcal{U}_q(\mathfrak{g})$-module.*

Proof. The proof of this statement is straightforward computation of the coefficients in relations above. We left it to the reader. □

2.2.1. *Quantum affine algebra $U(\widehat{\mathfrak{sl}_2})$.* Let us apply this construction to the Lie algebra $\mathfrak{sl}_2$. We have generators $e_r = x^{+}_{1,r}$, $f_r = x^{-}_{1,r}$ for $r \in \mathbb{Z}$, $h_m = h_{1,m}$ for $m \neq 0$, K and K^{-1}. Let us define power series $e(w)$, $f(w)$, $\phi^{\pm}(z)$ as in the general case. We have the following set of relations:

$$
\begin{aligned}
&KK^{-1} = K^{-1}K = 1,\\
&K\phi^{\pm}(z) = \phi^{\pm}(z)K,\\
&Ke = q^2 eK,\\
&Kf = q^{-2} fK,\\
&\phi^{\pm}(z)e(w) = \frac{q^2 w - z}{w - q^2 z} e(w)\phi^{\pm}(z),\\
&\phi^{\pm}(z)f(w) = \frac{q^{-2} w - z}{w - q^{-2} z} f(w)\phi^{\pm}(z),\\
&[e(z), f(w)] = \frac{\delta_{i,j}}{q - q^{-1}} \delta(\frac{w}{z})(\phi^{+}(w) - \phi^{-}(w)).
\end{aligned}
$$

2.3. **The quantum Heisenberg algebra.**

Definition 2.3.1. *The infinite-dimensional Heisenberg algebra $\mathcal{H}$ is the $\mathbb{C}$-algebra generated by elements $a_{\pm n}$ for $n \in \mathbb{Z}_{>0}$ and a central element γ with relations $[a_n, a_m] = \delta_{n,-m} n\gamma$.*

Let us fix complex numbers q_1, q_2 and $q = q_1 q_2$ and set

$$\alpha_k = \frac{(1 - q_1^k)(1 - q_2^k)(1 - q^{-k})}{n}.$$

We consider the algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$ generated over $\mathbb{C}(q_1, q_2)$ by elements $a_{\pm n}$ for $n \in \mathbb{Z}_{>0}$ and a central element c with relations $[a_n, a_m] = \delta_{n,-m}\frac{c^n - c^{-n}}{\alpha_n}$.

The algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$ is called the quantum Heisenberg algebra.

2.4. **The elliptic Hall algebra.** The main goal of this section is to construct an algebra $\mathcal{E}_K$ that, in a sense, is the quantum affinization of $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$ (note that $\widehat{\mathfrak{gl}_1}$ is not a Kac-Moody Lie algebra so the construction of the previous section does not apply literally but serves as motivation).
We set $Z = \mathbb{Z}^2$ and $Z^\times = \mathbb{Z}^2 \setminus (0,0)$. For an element $x = (a,b) \in Z^\times$ let us put $\deg(x) := g.c.d.(a,b) \in \mathbb{Z}_{>0}$. For a pair of elements $(x,y) \in (Z^2)^\times$ we set $\epsilon_{x,y} = sign(\det(x,y)) \in \{\pm 1\}$ and let $\Delta_{x,y}$ be the triangle in Z with vertices $(0,0), x, x+y$.

Definition 2.4.1. *The elliptic Hall algebra $\mathcal{E}_K$ is the $\mathbb{C}$ algebra generated by the set of elements u_x for $x \in Z^\times$ and κ_α for $\alpha \in Z$ subject to the relations given below. We define elements θ_z for $z \in Z^\times$ satisfying the following equations for every $x_0 \in Z^\times$ with $\deg(x_0) = 1$:*

$$\sum_i \theta_{ix_0} s^i = exp(\sum_{r \geq 1} \alpha_r u_{rx_0} s^r).$$

Note that $\theta_{x_0} = \alpha_1 u_{x_0}$, $\theta_{2x_0} = \alpha_2 u_{2x_0} + \frac{\alpha_1}{2} u_{x_0}^2$.
The generating relations of $\mathcal{E}_K$ are as follows.

$$\kappa_\alpha \text{ is central,}$$
$$\kappa_\alpha \kappa_\beta = \kappa_{\alpha+\beta},\ \kappa_0 = 1,$$
$$[u_x, u_y] = \delta_{x,-y} \frac{\kappa_x - \kappa_x^{-1}}{\alpha_{\deg(x)}} \text{ with } \deg((r,d)) := \gcd(r,d) \text{ if } x,y \text{ belong to the same line,}$$
$$[u_y, u_x] = \epsilon_{x,y} \kappa_{\alpha(x,y)} \frac{\theta_{x+y}}{\alpha_1} \text{ if } \deg(x) = 1 \text{ and } \Delta_{x,y} \text{ has no interior lattice point.}$$

In the expression above

$$\alpha(x,y) = \epsilon_x \frac{(\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x+y))}{2},\ \text{if } \epsilon_{x,y} = 1,$$
$$\alpha(x,y) = \epsilon_y \frac{(\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x+y))}{2},\ \text{if } \epsilon_{x,y} = -1.$$

Here $\epsilon_x = 1$ for $x = (r,d)$ if $r > 0$ or $r = 0$ and $d > 0$ and $\epsilon_x = -1$ in other case.

For any line L through the origin with a rational slope elements u_x for $x \in L$ satisfy relations of the quantum Heisenberg algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$. Therefore $\mathcal{E}$ can be understood as the quantum affinization of the quantum Heisenberg algebra.
Note that all κ_z are defined from $\kappa_{0,1}$ and $\kappa_{1,0}$. In this talk we will be interested in specialization $\mathcal{E}$ of this algebra to the case $\kappa_{0,1} = 1$, $\kappa_{1,0} = c$.

Corollary 2.4.2. *The algebra $\mathcal{E}$ (we will also call it the EHA) is the $\mathbb{C}(c^{\pm}1)$ algebra generated by the set of elements u_x for $x \in Z^\times$ subject to the following relations:*

$$(i)[u_x, u_{x'}] = 0 \text{ if } x, x' \text{ belong to the same line and } x \neq -x',$$
$$(i')[u_x, u_{x'}] = \frac{c^r - c^{-r}}{q - q^{-1}} \text{ if } x = (r,d), x' = (-r,-d),$$
$$(ii)[u_y, u_x] = \epsilon_{x,y} \frac{\theta_{x+y}}{\alpha_1} \text{ if } \deg(x) = 1, \epsilon_x = \epsilon_y \text{ and } \Delta_{x,y} \text{ has no interior lattice point,}$$
$$(ii')[u_y, u_x] = \epsilon_{x,y} \frac{c^{\alpha(r,r')}}{\alpha_1} \theta_{x+y} \text{ if } \deg(x) = 1, x = (r,d), y = (r',d'), \epsilon_x \neq \epsilon_y \text{ and } \Delta_{x,y} \text{ has no interior lattice point.}$$

Let us give few examples of commutation relations:

$$
\begin{aligned}
&[u_{(1,0)}, u_{(0,1)}] = \frac{\theta_{(1,1)}}{\alpha_1} = u_{(1,1)},\\
&[u_{(1,0)}, u_{(1,2)}] = \frac{\theta_{(2,2)}}{\alpha_1} = \frac{\alpha_2}{\alpha_1} u_{(2,2)} + \frac{\alpha_1}{2} u_{(1,1)},\\
&[u_{(-1,0)}, u_{(1,2)}] = -\frac{\kappa_{1,0}}{\alpha_1}\theta_{(2,2)} = -\frac{\alpha_2 c}{\alpha_1} u_{(2,2)} - \frac{\alpha_1 c}{2} u_{(1,1)},
\end{aligned}
$$

$[u_{(1,2)}, u_{(1,-1)}]$ is not proportional to $u_{(2,1)}$. This commutator will be computed later.

3. Properties of EHA

3.1. $\mathrm{SL}(2,\mathbb{Z})$**-action.** We have a natural action of $\mathrm{SL}(2,\mathbb{Z})$ on the generators u_x of the algebra $\mathcal{E}$. An element $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ sends u_x to $u_{\gamma(x)}$. If $c \neq 1$ this action does not preserve $\alpha(x,y)$ and therefore does not descend to $\mathcal{E}$. In fact we have an action of the universal cover $\widehat{\mathrm{SL}}(2,\mathbb{Z})$ on $\mathcal{E}$. Nevertheless it sends θ_x to $\theta_{\gamma(x)}$, $\epsilon_{x,y}$ to $\epsilon_{\gamma(x),\gamma(y)}$, preserves degrees and triangles w/o interior lattice points. Suppose that x, y satisfy condition of the commutation relation (iii). Note that assuming $c = 1$ we have the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{E}$. Therefore if x, $\gamma(x)$, y and $\gamma(y)$ lie the right half-plane (so $\alpha(x,y) = 0$), then the commutation relation is preserved by the action of γ. We will use it in Section 3.4.

3.2. **Smaller set of generators.** For any element $x = (a,b) \in Z^\times$ we define its rank as $\mathrm{rank}(x) := a$.

Lemma 3.2.1. *The EHA $\mathcal{E}_K$ (ansd therefore its specialization $\mathcal{E}$) is generated by the elements $u_{\pm1,l}$, $u_{0,\pm k}$ for $l \in \mathbb{Z}$, $k \in \mathbb{Z}_{>0}$.*

Proof. We will prove this lemma by the induction on the rank of an element. Denote by $\mathcal{T}$ the subalgebra generated by $u_{\pm1,l}$, $u_{0,\pm k}$ and assume that $u_{r,s} \in \mathcal{T}$ for any $(r,s) \in Z^\times$ with $n > |r| \geq 1$. It is enough to prove that $u_{n,d} \in \mathcal{T}$, the fact for the $u_{-n,d}$ will follow analogously. Let us denote $z = (n,d)$ and define $x = (r,s)$ be the closest point to the line $0z$ such that $r < n$. By the construction $\Delta_{x,z-x}$ has no interior points (they need to be closer to the line $0z$ then x) and $\epsilon_x = \epsilon_{z-x} = \epsilon_z = 1$. Therefore $\theta_z = \frac{1}{\alpha_1}[t_x, t_{z-x}]$ and ranks of x and $z-x$ are less then n. Therefore $\theta_z \in \mathcal{T}$. Let $z = kz_0$ where $\deg(z_0) = 1$. From the definition $\theta_z = \alpha_k u_z + \sum_{i_1+\ldots+i_m=k} \beta_{i_1,\ldots,i_m} u_{i_1 z_0} \ldots u_{i_m z_0}$ where $\beta_{i_1,\ldots,i_m} = \frac{\prod_{j=1}^m \alpha_{i_j}}{m!}$. Every summand except for $\alpha_k u_z$ belongs to $\mathcal{T}$, so $u_z \in \mathcal{T}$. □

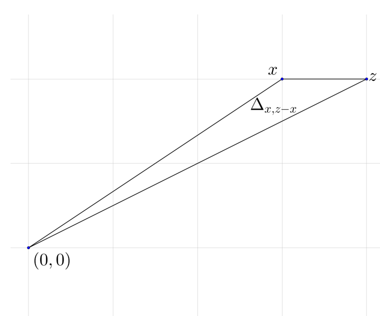

Figure 1. The induction step.

3.3. **Basis combinatorial notions.** In the next section we will give a basis of the EHA $\mathcal{E}$ as a vector space. For this purpose we need to introduce more notations.
For an element $z \in Z^\times$ we define its slope $\mu(z)$ to be the angle between the horizontal axis and the ray $0z$. We set $\mu(z) \in]-\frac{\pi}{2}, \frac{3\pi}{2}]$.

For the every sequence $s = (x_1, x_2, \ldots, x_n)$ of elements in $Z^\times$ we associate a broken line in Z connecting points $0, x_1, x_1 + x_2, \ldots, x_1 + \ldots + x_n$. We call two sequences s and $s^{'}$ equivalent if $s^{'}$ can be obtained from s by successive permutations of adjacent vertices $x_i \neq -x_{i+1}$ of the same slope. We will refer to equivalence classes of sequences as paths and denote the set of all paths as Path. To a path $p = (x_1, x_2, \ldots, x_n)$ we assign the element $u_p = u_{x_1} u_{x_2} \ldots u_{x_n} \in \mathcal{E}$. From the definition of $\mathcal{E}$ we see that the elements u_p generate EHA as a vector space.

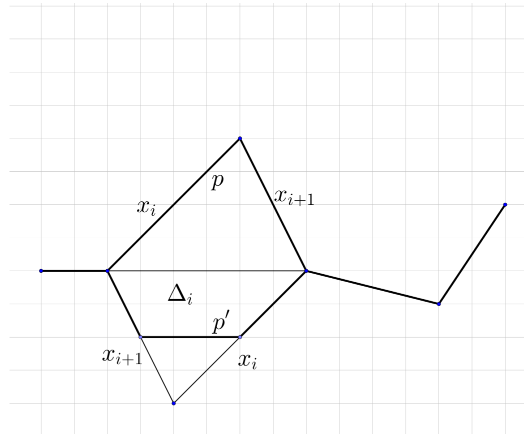

FIGURE 2. The triangle Δ and a local convexification p' of the non-convex path p.

We say that the path p represented by a sequence $(x_1, x_2, \ldots, x_n)$ is convex if it satisfies

$$-\frac{\pi}{2} < \mu(x_1) \leq \mu(x_2) \leq \ldots \leq \mu(x_n) \leq \frac{3\pi}{2}.$$

We denote the set of all convex paths as Conv. For example the path p in Figure 2 is not convex because $\mu(x_i) > \mu(x_2)$ but the path p' is convex. In next section we will prove that u_p for $p \in \mathsf{Conv}$ is a basis of $\mathcal{E}$.

For the pair x_i, x_{i+1} in a path p such that $\mu(x_i) > \mu(x_{i+1})$ we can construct the triangle Δ with vertices $x_1 + \ldots + x_{i-1}, x_1 + \ldots + x_{i-1} + x_{i+1}, x_1 + \ldots + x_{i-1} + x_i + x_{i+1}$. We call the path p' obtained from p by replacing x_i, x_{i+1} by a convex path in the triangle Δ_i *a local convexification* of p.

For the path $p = (x_1, x_2, \ldots, x_n)$ we define its *length* $l(p) = n$ and *weight* $|p| = \sum_{i=1}^n x_i$. Note that for every path p there is a unique convex path $p^\sharp$ with the same entries that in p. This path is constructed by permuting entries according to the order given by slope. Note that any two segments of paths p and $p^\sharp$ either do not intersect or coincide. Let us consider the subalgebra $\mathcal{E}^+$ of $\mathcal{E}$ generated by $u_{(a,b)}$ with $a > 0$ or $a = 0$, $b > 0$. It is generated by paths with entries $x_i = (a_i, b_i)$, $a_i > 0$ or $a_i = 0$, $b_i > 0$ for all i. We call them positive paths. Let us denote Conv^+ the set of positive convex paths. We can define the subalgebra $\mathcal{E}^-$ and the subset Conv^- in the same way. Then any positive p and the corresponding positive convex path $p^\sharp$ bound a polygon with all vertices in lattice. Let $a(p)$ to be the area of this polygon, we will abuse the terminology and call $a(p)$ the area of the path p. We have the following lemma.

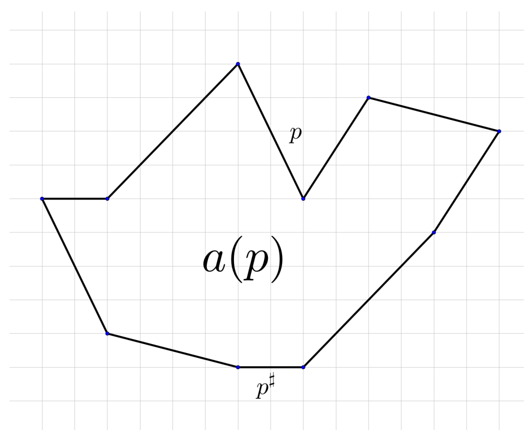

FIGURE 3. The area $a(p)$ of the path p.

Lemma 3.3.1. *Let p be a positive path in $Z^\times$. Then*
i) p is convex if and only if $a(p) = 0$,
ii) for any subpath p' of p we have $a(p') \leq a(p)$,
iii) For any local convexification p' of p we have $a(p') < a(p)$.

i) is obvious.
To prove ii) let us consider the path p^* obtained from p by replacing p' by $p'^\sharp$. Note that $p^{*,\sharp} = p^\sharp$ because p^* is obtained by permuting entries. Therefore $a(p) = a(p') + a(p^*) \Rightarrow a(p') \leq a(p)$.
So it remains to prove the third statement of the lemma.

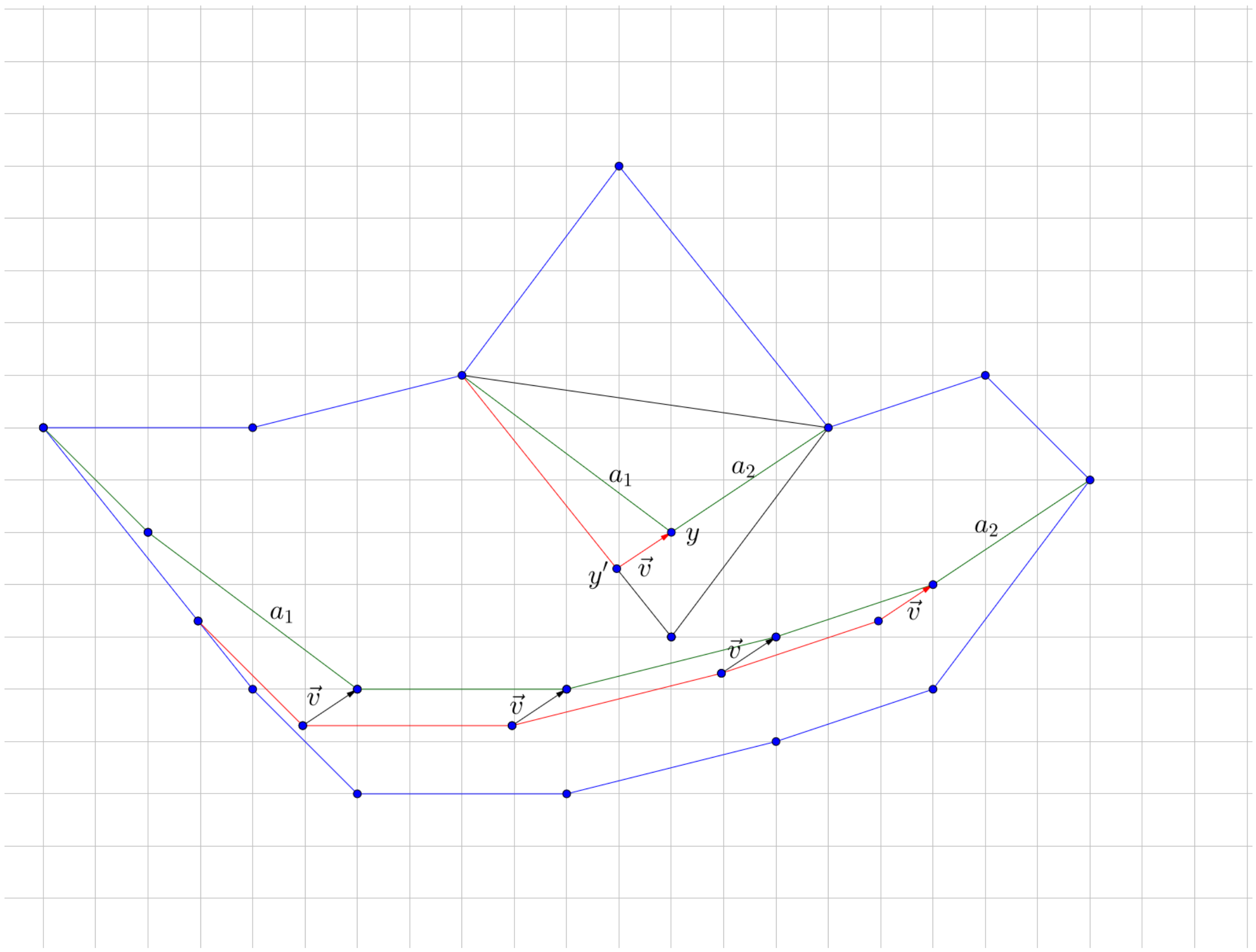

FIGURE 4. The lower green path $p'^\sharp$ lies between upper green p' and red $p''^\sharp$.

Let p' be a local convexification of p obtained by replacing entries x_i, x_{i+1} by a convex path. We claim that $p'^\sharp$ belongs to the polygon bounded by $p^\sharp$ and p'. The upper bound is obvious. Suppose that we have points inside the triangle Δ for p'. Let $y \in p'$ be the first such vertex and let p'' be a broken line defined in the following way. We take the edge from y and extend it in the direction of y to the intersection y' with the edge of Δ. We replace y by y' in the path p'. The broken line p'' doesn't have to be a path but we still can define notions of slope, convex broken line and area in the same way. We claim that $p'^\sharp$ belongs to the polygon A_y bounded by p' and $p''^\sharp$. Indeed let us look Figure 4. Here blue color corresponds to the path p and corresponding convex path $p^\sharp$, green color – to p' and $p'^\sharp$ and red – to p'' and $p''^\sharp$. Note that we are interested only in z with $\mu(x_{i+1}) \leq \mu(z) \leq \mu(x_i)$. Let us introduce some notations. We denote the incoming in y segment of p' by a_1 and the outgoing segment by a_2. Therefore for the first edges in $p'^\sharp$ and $p''^\sharp$ we have that the slope of the second one is bigger. Therefore up to the segment a_1 we have $p'^\sharp$ belongs to A_y. Let us denote by $\vec{v}$ the vector between y' and y. Note that we have this vector (black color on the picture) between corresponding vertices of $p'^\sharp$ and $p''^\sharp$ after the segment a_1 in $p'^\sharp$. After that we have the same segments in both of broken lines up to the segment a_2. And we have $p'^\sharp$ and $p''^\sharp$ coincide after a_2. Therefore $p'^\sharp$ belongs to A_y.
So it is enough to prove the claim for the convexification p' without interior lattice points. The same argument applied to p' gives that it is enough to prove for the case when a point on a side of Δ should be the intersection of it's sides, i.e. the vertex. In this case we get $p'^\sharp = p^\sharp$.

Let us denote by B the polygon bounded by p and p'. Therefore

$$a(p) \geq a(p') + S(B) \geq a(p') + s(\Delta) \Rightarrow a(p') < a(p).$$

3.4. **Basis of convex paths.** The main goal of this section is to prove an important lemma that u_p for $p \in \mathsf{Conv}^+$ give a basis of $\mathcal{E}^+$ as a vector space. In future talks we will show from another description of the EHA that elements corresponded to convex paths are linearly independent. For now we refer reader for the proof to the paper [BS] of Burban and Shiffmann. So it remains to check that these elements span everything. For these purposes we need to recall a standard fact about area of the polygon with lattice points.

Proposition 3.4.1. ***(Pick's formula.)*** *For the polygon with lattice points P with $i(P)$ interior lattice points and $b(P)$ lattice points on the boundary we have the following formula*

$$S(P) = i(P) + \frac{b(P)}{2} - 1$$

The most important step to prove that convex paths span the whole EHA is to check that every non-convex path of length 2 is generated by convex paths. Let us show that the statement for an arbitrary path p will follow.

Lemma 3.4.2. *Suppose that $[u_x, u_y] \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$ for any x, y such that $|\det(x,y)| < d$. Then for any positive path p satisfying $a(p) < d$ we have $u_p \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$*

Proof. For $a(p) = 0$, Lemma 3.3.1 i) states that p is convex, so proposition holds. If $a(p) > 0$ then p is not convex, so we have $\mu(x_1) \leq \mu(x_2) \leq \ldots \leq \mu(x_s) > \mu(x_{s+1})$ for some s. The statement ii) of Lemma 3.3.1 gives us that $det(x_s, x_{s+1}) = a((x_s, x_{s+1})) \leq a(p) < d$, so by the lemma assumption $u_{x_s}u_{x_{s+1}} = \sum_i \beta_i u_{q_i}$ where q_i is a local convexification of p. Therefore it is enough to prove that $u_{q_i} \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$ for every i. By Lemma 3.3.1 iii) we have $a(q_i) < a(p)$. Note that the area function $a(\bullet)$ takes values of the form $\frac{n}{2}$ for $n \in \mathbb{Z}_{\geq 0}$ and $a(q) < a(p) < d$ for every convexification q. Applying the same procedure to q_i finitely many times we will get a linear combination of convex paths. The lemma follows. $\square$

Ir remains to show that $[u_x, u_y] \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$ for any two segments $x, y \in \mathbb{Z}^\times$.

Proposition 3.4.3. *For any elements $x, y \in Z^\times$ with $\mu(x) > \mu(y)$ we have $u_x u_y \in \bigoplus_{p \in I_{x,y}} \mathbb{C}u_p$ where $I_{x,y}$ – the set of convex paths inside the triangle $\Delta_{x,y}$.*

Proof. We will prove this lemma by the induction on $\det(y,x)$. If $\det(y,x) = 1$ then by Pick's formula for $S(\Delta_{x,y}) = \frac{1}{2}$ we have $i(\Delta_{x,y}) = b(\Delta_{x,y}) = 0$, so $\deg(x) = \deg(x+y) = 1$ and $\Delta_{x,y}$ has no interior lattice points. It follows that $u_x u_y = u_y u_x + [u_x, u_y] = u_y u_x + \kappa_{\alpha(x,y)} u_{x+y}$.
Let us assume that the statement of the lemma holds for any x', y' with $\det(x', y') < d$ (if $\det(y', x') < 0$ then the path (x', y') is convex) and set $\det(y,x) = d$. Let us first consider the case when $\Delta_{x,y}$ has no interior lattice points.

Lemma 3.4.4. *If $\Delta_{x,y}$ has no interior lattice points then $u_x u_y \in \bigoplus_{p \in I_{x,y}} \mathbb{C}u_p$.*

Proof. We put $y_0 = \frac{y}{\deg(y)}$, $x_0 = \frac{x}{\deg(x)}$ and $(x+y)_0 = \frac{x+y}{\deg(x+y)}$.
1) Assume $\deg(x) \geq 2$, $\deg(y) \geq 2$
Let us consider the point $z = (\deg(x)-1)x_0 + y_0 \in \Delta_{x,y}$. The only case when z is not an interior point is $\deg(x) = \deg(y) = 2$. We can find such $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ that $x = \gamma((0,2))$ and $y = \gamma((2,0))$. As both points x and y are positive action of γ preserves commutation relations. Direct computation shows

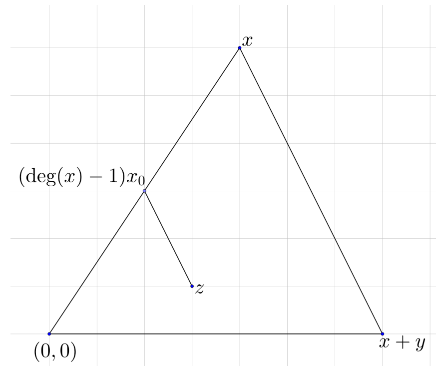

FIGURE 5. The interior lattice point z for case 1).

$$u_{(2,0)} = \frac{\alpha_1}{\alpha_2}[u_{(1,1)}, u_{(1,-1)}] - \frac{\alpha_1^2}{2}u_{(1,0)}^2 \Rightarrow$$

$$[u_{(0,2)}, u_{(2,0)}] = \frac{\alpha_1}{\alpha_2}[u_{(0,2)}, [u_{(1,1)}, u_{(1,-1)}]] - \frac{\alpha_1^2}{2}[u_{(0,2)}, u_{(1,0)}^2]$$

It is enough to check that each summand can be decomposed into a linear combination of monomials corresponding to convex paths. Indeed

$$[u_{(0,2)}, [u_{(1,1)}, u_{(1,-1)}]] =$$
$$[u_{(1,1)}, [u_{(0,2)}, u_{(1,-1)}]] + [u_{(1,-1)}, [u_{(1,1)}, u_{(0,2)}]] =$$
$$-[u_{(1,1)}, u_{(1,1)}] + [u_{(1,-1)}, u_{(1,3)}] = [u_{(1,-1)}, u_{(1,3)}] = \frac{1}{\alpha_1}\theta_{(2,2)}$$

$$[u_{(0,2)}, u_{(1,0)}^2] = [u_{(0,2)}, u_{(1,0)}]u_{(1,0)} + u_{(1,0)}[u_{(0,2)}, u_{(1,0)}] = -u_{(1,2)}u_{(1,0)} - u_{(1,0)}u_{(1,2)} =$$
$$-[u_{(1,2)}, u_{(1,0)}] + 2u_{(1,0)}u_{(1,2} = -\frac{1}{\alpha_1}\theta_{(2,2)} + 2u_{(1,0)}u_{(1,2)} \in \mathbb{C}u_{(2,2)} \oplus \mathbb{C}u_{(1,1)} \oplus \mathbb{C}u_{(1,0)}u_{(1,2)}.$$

Therefore $u_x u_y = u_y u_x + \gamma([u_{(0,2)}, u_{(2,0)}]) \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$.

2) Suppose $\deg(x) = 1$ or $\deg(y) = 1$. Then

$$u_x u_y = u_y u_x + \frac{\kappa_{\alpha(x,y)}\theta_{x+y}}{\alpha_1} =$$
$$u_y u_x + \kappa_{\alpha(x,y)} \sum_{i_1+\ldots+i_m = \deg(x+y)} \beta_{i_1,\ldots,i_m} u_{i_1(x+y)_0} \ldots u_{i_m(x+y)_0} \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q.$$

∎

Therefore we may assume that there are interior points in $\Delta_{x,y}$. Let $z \in \Delta_{x,y}$ be a point such that the triangle $0xz$ has no interior points and $\deg(z) = \deg(x-z) = 1$. Suppose that $x-z$ is positive. The case when $x-z$ is negative is similar, we will have same expressions but with coefficients depending on κ. We have

$$[u_z, u_{x-z}] = \frac{\theta(x)}{\alpha_1} = \frac{\alpha_{\deg(x)}}{\alpha_1} u_x + f,$$

where f is generated by elements u_{kx_0} for $f < \deg(x)$. We get

$$[u_x, u_y] = \frac{\alpha_1}{\alpha_{\deg(x)}} [[u_z, u_{x-z}], u_y] - [f, u_y] =$$

$$\frac{\alpha_1}{\alpha_{\deg(x)}} [[u_y, u_{x-z}], u_z] - \frac{\alpha_1}{\alpha_{\deg(x)}} [[u_y, u_z], u_{x-z}] - [f, u_y].$$

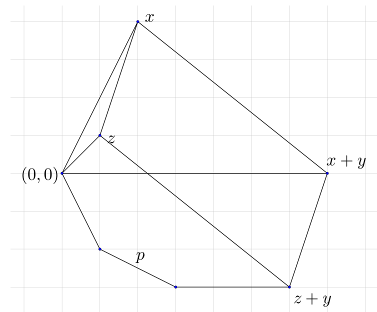

FIGURE 6. The case of interior lattice point $z \in \Delta_{x,y}$.

We see that the triangles $0zy$ and xyz lie inside the triangle $0xy$. Therefore we have $\det(y,z) = 2S(0zy) < 2S(0xy) = \det(y,x)$ and analogously $\det(y, x-z) < \det(y,x)$. By the induction hypothesis we have $[u_y, u_z] \in \bigoplus_{p \in I_{z,y}} \mathbb{C}u_p$ and $[u_y, u_{z-x}] \in \bigoplus_{q \in I_{z-x,y}} \mathbb{C}u_q$.
From the construction $\mu(x-z) < \mu(z) < \mu(y)$, so $(x-z, p)$ is a convex path for all $p \in I_{z,y}$. Therefore the path $(p, x-z)$ is a local convexification of the path $(x, y, x-z)$. By Lemma 3.3.1 iii) we have

$a((p, x-z)) < a((z, y, x-z)) < a(x,y) = \det(y,x) = d$. So $[u_{x-z}, u_p] \in \bigoplus_{t \in I_{x,y}} \mathbb{C}u_t$.

Analogously $[u_q, u_x] \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$. Therefore it is enough to prove that $[f, u_y] \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$. Note that f is a finite sum of paths $u_{i_1x_0}u_{i_2x_0}\ldots u_{i_mx_0}$ with some coefficients, where $i_1 + \ldots + i_m = \deg(x)$. We need to show that $[u_{i_1x_0}u_{i_2x_0}\ldots u_{i_mx_0}, u_y] \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$. Let us state a stronger fact:

Lemma 3.4.5. *Let $p = (i_1x_0, i_2x_0, \ldots, i_mx_0)$ be a positive path with $i_1 + \ldots + i_m \le \deg(x)$ and $i_j < \deg(x)$ for all j and q be a convex path between $(i_1 + \ldots + i_m)x_0$ and y. Then $u_pu_q \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$.*

Proof. We prove the proposition by induction on m. If $m = 1$ let us denote $x' = i_1x_0$. Then q is a local convexification of the path $(x', y-x')$, so $a((p,q)) < a((x', y-x')) < a((x,y)) = det(x,y) = d$, so the statement follows from Proposition 3.4.3. Here the first inequality is given by Lemma 3.3.1 iii) and the second one follows from the fact that the triangle $0x'y$ is contained in $0xy$.
Suppose that the proposition holds for all paths p with $k < m$ segments. We have

$$[u_p, u_q] = [u_{i_1x_0}u_{i_2x_0}\ldots u_{i_mx_0}, u_q] = [u_{i_1x_0}u_{i_2x_0}\ldots u_{i_{m-1}x_0}, u_q]u_{i_mx_0} + u_{i_1x_0}\ldots u_{i_{m-1}x_0}[u_{i_mx_0}, u_q]$$

By the induction hypothesis the first summand of the right hand side is a linear combination of the elements $u_tu_{i_mx_0}$ where t is a convex path with all slopes between $\mu(y)$ and $\mu(x)$. Therefore $(t, i_mx_0) \in I_{x,y}$.
The second summand by the proposition for $m = 1$ is a linear combination of the elements $u_{i_1x_0}u_{i_2x_0}\ldots u_{i_{m-1}x_0}u_s$ where s is a convex path with all slopes between $\mu(y)$ and $\mu(x)$. By the induction hypothesis it can be rewritten as a linear combination of u_r for $r \in I_{x,y}$. Therefore $[u_p, u_q] \in \bigoplus_{r \in I_{x,y}} \mathbb{C}u_r$, q.e.d. $\square$

Applying Lemma 3.4.5 above to $q = (y)$ and all summands in f we get $[f, x] \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$. Therefore the sum $[u_x, u_y] = \frac{\alpha_1}{\alpha_{\deg(x)}}[[u_y, u_{x-z}], u_z] - \frac{\alpha_1}{\alpha_{\deg(x)}}[[u_y, u_z], u_{x-z}] - [f, u_y] \in \bigoplus_{q \in I_{x,y}} \mathbb{C}u_q$, q.e.d. $\square$

Proposition 3.4.6. *The algebra $\mathcal{E}^+$ is isomorphic to $\bigoplus_{p \in \mathit{Conv}^+} \mathbb{C}u_p$ as a vector space.*

Proof. We have already said that elements u_p are linearly independent for convex paths p. Let us consider a non-convex path $q = (x_1, x_2, \ldots, x_n)$ and let $\mu(x_1) \leq \ldots \mu(x_s) > \mu(x_{s+1})$. Then by Proposition 3.4.3 $u_q = \sum_i \beta_i u_{q_i}$ where q_i is a local convexification of q. By Lemma 3.3.1 we have $a(q_i) < a(q)$. We can apply the same procedure to q_i. The area function takes only half-integer non-negative values, so in a finite number of steps we get $u_q \in \mathbb{C}u_p{}_{p \in \mathsf{Conv}^+}$. $\square$

Example 3.4.7. *Let us show the decomposition of $[u_{(1,2)}, u_{(1,-1)}]$ into a linear combination of elements corresponding to convex paths.*

$$u_{(1,-1)} = \frac{1}{\kappa_{0,-1}}[u_{(0,-1)}, u_{(1,0)}] = [u_{(0,-1)}, u_{(1,0)}],$$

$$[u_{(1,2)}, u_{(1,-1)}] = [u_{(1,2)}, [u_{(0,-1)}, u_{(1,0)}]] = [u_{(0,-1)}, [u_{(1,2)}, u_{(1,0)}]] + [u_{(1,0)}, [u_{(0,-1)}, u_{(1,2)}]] =$$

$$[u_{(0,-1)}, \frac{\theta_{(2,2)}}{\alpha_1}] + [u_{(0,1)}, u_{(1,1)}] = u_{(2,1)} - \frac{\alpha_1}{2}[u_{(0,-1)}, u^2_{(1,1)}] - \frac{\alpha_2}{\alpha_1}[u_{(0,-1)}, u_{(2,2)}] =$$

$$\frac{\alpha_1 - \alpha_2}{\alpha_1} u_{2,1} + \frac{\alpha_1}{2}(u_{(1,1)}u_{(1,0)} + u_{(1,0)}u_{(1,1)}) =$$

$$= \frac{\alpha_1 - \alpha_2}{\alpha_1} u_{(2,1)} + \alpha_1 u_{(1,0)}u_{(1,1)} + \frac{\alpha_1}{2}[u_{(1,1)}, u_{(1,0)}] = \frac{\alpha_1 - 2\alpha_2}{2\alpha_1} u_{2,1} + \alpha_1 u_{(1,0)}u_{(1,1)}.$$

3.5. **Triangular decomposition of the EHA.** Let us denote by $\mathcal{E}^>$, $\mathcal{E}^<$ and $\mathcal{E}^0$ subalgebras generated by $u_{1,l}$, $u_{-1,l}$ and $u_{0,\pm k}$ respectively. Note that $\mathcal{E}^> \not\simeq \mathcal{E}^+$ cause we have not elements $u_{0,m}$ in $\mathcal{E}^>$. We have the following important corollary of Proposition 3.4.6.

Proposition 3.5.1. *The EHA $\mathcal{E}$ has a triangular decomposition $\mathcal{E}^> \otimes \mathcal{E}^0 \otimes \mathcal{E}^< \simeq \mathcal{E}$ where the isomorphism is given by the multiplication map.*

Proof. First, we need to check that the multiplication map $m : \mathcal{E}^> \otimes \mathcal{E}^0 \otimes \mathcal{E}^< \to \mathcal{E}$ is surjective. We have the following set of relations

$$(i)[u_{(1,l)}, u_{(0,m)}] = u_{(1,l+m)} \in \mathcal{E}^>, \ [u_{(1,l)}, u_{(0,-m)}] = -\kappa_{0,-m}u_{(1,l+m)} = -u_{(1,l+m)} \in \mathcal{E}^>,$$

$$(ii)[u_{(-1,l)}, u_{(0,m)}] = -\kappa_{0,m}u_{(-1,l+m)} = -u_{(-1,l+m)} \in \mathcal{E}^<, \ [u_{(-1,l)}, u_{(0,-m)}] = u_{(-1,l-m)} \in \mathcal{E}^<,$$

$$(iii)[u_{(1,l)}, u_{(-1,m)}] = \pm\frac{\theta_{(0,l+m)}}{\alpha_1} \in \mathcal{E}^0 \text{ if } l \neq -m,$$

$$(iv)[u_{(1,l)}, u_{(-1,-l)}] = \frac{\kappa_{1,l} - \kappa_{-1,-l}}{\alpha_1} = \frac{c - c^{-1}}{\alpha_1},$$

$$(v)[u_{(0,\pm l)}, u_{(0,\pm m)}] = 0.$$

Therefore in the element u_p for $p = (x_1, \ldots, x_n)$ we can move all $x_i \in \mathcal{E}^+$ to the beginning using relations (i), (iii) and (iv) and get a linear combination of elements u_{p_i} for paths $p_i = (q_i, y_1, \ldots, y_k)$ where $u_{q_i} \in \mathcal{E}^>$ and all $y_i \in \mathcal{E}^<$ or $y_i \in \mathcal{E}^0$. In the same way using relations (ii) we can move all $x_i \in \mathcal{E}^<$ to the end and get a linear combination of the elements u_{s_i} fo $s_i = (q_i, t_i, r_i)$ where $u_{q_i} \in \mathcal{E}^>$, $u_{t_i} \in \mathcal{E}^0$ and $u_{r_i} \in \mathcal{E}^<$. But all such u_{s_i} belong to the image of m. Therefore m is surjective.

By Proposition 3.4.6 convex paths form a basis in both $\mathcal{E}^{>}$ and $\mathcal{E}^{<}$ (the proof is analogous). The basis of $\mathcal{E}^0$ is given by all paths with all segments of form $(0,l)$. We denote the corresponding sets of convex paths $\mathsf{Conv}^{>}$, Conv^0 and $\mathsf{Conv}^{<}$. Therefore it is enough to prove that $m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})$ are linearly independent in $\mathcal{E}$ for all elements $v_i\in\mathsf{Conv}^{>}$, $t_j\in\mathsf{Conv}^0$ and $w_k\in\mathsf{Conv}^{<}$. Let us consider two cases.

1) t_j is a path consisting of entries $(0,n)$ for $n\in\mathbb{Z}_{>0}$. Note that for any elements $u_{(r,d)}$ with $r>0$, $u_{(0,n)}$ with $n>0$ and $u_{(r',d')}$ with $r'<0$ we have $\mu((r,d))<\mu((0,n))<\mu((r',d'))$. Therefore $m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})$ are convex paths that are linearly independent by analogous to Proposition 3.4.6 result for $\mathcal{E}$.

2) Suppose that t_j has an entry $(0,-n)$. Using relations (v) we can set $t_j=((0,a_1),(0,a_2),\dots,(0,a_k)$ with $a_1<a_2<\dots<a_k$. Let us denote the subalgebra generated by $u_{(0,n)}$ (resp. $u_{(0,-n)}$) as $\mathcal{E}^{0,+}$ (resp, $\mathcal{E}^{0,-}$). We set $u_{t_j^+}=\prod_{a_i>0}u_{(0,a_i)}\in\mathcal{E}^{0,+}$ and $u_{t_j^-}=\prod_{a_i<0}u_{(0,a_i)}\in\mathcal{E}^{0,-}$. Let us show that the multiplication map $m':\mathcal{E}^{<}\otimes\mathcal{E}^{0,-}\to\mathcal{E}$ is injective.
We call a path $p=(x_1,x_2,\dots,x_r)$ concave if $(x_r,\dots,x_2,x_1)$ is convex. Analogously to Proposition 3.4.6 we can show that elements corresponding to concave paths form a basis in $\mathcal{E}$ and $\mathcal{E}^{<}$. Then if we choose the concave basis u_p in $\mathcal{E}^{<}$ and a basis u_q in $\mathcal{E}^{0,-}$ then $m'(u_p\otimes u_q)$ will be elements corresponding to different concave paths and therefore m' is injective.
Let $\sum_q u_q$ be the decomposition of an element $m(u_{v_i}\otimes u_{t_j^-})$ into a linear combination of convex paths. Let p' be a local convexification of a path p that replaces entries x_i,x_{i+1} by $y_1,\dots,y_k$. Proposition 3.4.3 implies that $\mu(x_i)>\mu(y_j)>\mu(x_{i+1})$ for all j. Therefore $m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})=\sum_q m(u_q\otimes u_{t_j^+}\otimes u_{w_k})$ – linear combination of convex paths. Let Conv_{t_j,w_k} be a set of convex paths p that have form $p=(x_1,x_2,\dots,x_l,t_j,w_k)$ with $\mu(x_i)>\frac{\pi}{2}$, $\forall i$. (We allow l to be 0.) Note that $\mathsf{Conv}=\bigsqcup\limits_{t_j,w_k}\mathsf{Conv}_{t_j,w_k}$. Injectivity of m' implies that all $m'(u_{v_i}\otimes u_{t_j^-})$ ae linearly independent. Therefore $m(m'(u_{v_i}\otimes u_{t_j^-})\otimes u_{t_j^+}\otimes u_{w_k})=m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})\in\mathsf{Conv}_{t_j,w_k}$ are linearly independent. It follows that m is injective. □

4. Ding-Iohara algebra

4.1. **Generators and relations.** We give explicit generators and relations for the Ding-Iohara algebra $\mathcal{U}$. Let us fix complex numbers q_1, q_2 and $q=q_1q_2$. Recall that $\alpha_k=(1-q_1^k)(1-q_2^k)(1-q^{-k})$. We set

$$\chi(z,w)=(z-q_1w)(z-q_2w)(z-q^{-1}w),$$
$$\delta(\frac{z}{w})=\sum_{k\in\mathbb{Z}}(\frac{z}{w})^k.$$

We will define an algebra $\tilde{\mathcal{U}}$ by generators $e_k,f_k,h_n^{\pm}$ where $k\in\mathbb{Z}$ and $n\in\mathbb{Z}_{>0}$ and relations to be specified below.
Let us define generating series

$$e(z):=\sum_{k\in\mathbb{Z}}e_kz^{-k},$$
$$f(z):=\sum_{k\in\mathbb{Z}}f_kz^{-k},$$
$$\psi^{\pm}(z)=1+\sum_{n\in\mathbb{Z}_{>0}}h_n^{\pm}z^{\pm n}.$$

Let ϵ, ϵ_1 and ϵ_2 be elements of $\{1,-1\}$. The defining relations of $\tilde{\mathcal{U}}$ are as follows.

$$\begin{aligned}
&\text{(i) } \psi^{\epsilon_1}(z)\psi^{\epsilon_2}(w) = \psi^{\epsilon_2}(w)\psi^{\epsilon_1}(z),\\
&\text{(ii) } \chi(z,w)\psi^{\epsilon_1}(z)e(w) = -\chi(w,z)e(w)\psi^{\epsilon_1}(z),\\
&\text{(iii) } \chi(z,w)e(z)e(w) = -\chi(w,z)e(w)e(z),\\
&\text{(iv) } \chi(w,z)f(z)f(w) = -\chi(z,w)f(w)f(z),\\
&\text{(v) } [f(z),e(w)] = \frac{1}{\alpha_1}\left(\delta(\frac{z}{w})(c\psi^-(z) - c^{-1}\psi^+(z)\right).
\end{aligned}$$

We are interested in its quotient $\tilde{\mathcal{E}}$ by cubic relations. We put $\operatorname*{Res}_{zyw}(a)$ be the coefficient of $(zyw)^{-1}$ in a. Cubic relations are as follows.

$$\operatorname*{Res}_{zyw}\left[(zyw)^m(z+w)(y^2-zw)e(z)e(y)e(w)\right] = 0 \text{ for all } m\in\mathbb{Z},$$
$$\operatorname*{Res}_{zyw}\left[(zyw)^m(z+w)(y^2-zw)f(z)f(y)f(w)\right] = 0 \text{ for all } m\in\mathbb{Z}.$$

The algebra $\tilde{\mathcal{E}}$ is called the quantum toroidal $\mathfrak{gl}_1$.
It will be useful to rewrite last two relations. We denote the coefficient of $z_1^{l_1}z_2^{l_2}z_3^{l_3}$ in a as $\underset{z_1^{l_1}z_2^{l_2}z_3^{l_3}}{C}(a)$.

$$\begin{aligned}
&\operatorname*{Res}_{zyw}\left[(zyw)^{-m}(z+w)(y^2-zw)e(z)e(y)e(w)\right] = \underset{(zyw)^{m-1}}{C}\left[(z+w)(y^2-zw)e(z)e(y)e(w)\right] =\\
&\underset{z^my^{m+1}w^{m-1}}{C}[e(z)e(y)e(w)] - \underset{z^{m+1}y^{m-1}w^m}{C}[e(z)e(y)e(w)] + \underset{z^{m-1}y^{m+1}w^m}{C}[e(z)e(y)e(w)] -\\
&\underset{z^my^{m-1}w^{m+1}}{C}[e(z)e(y)e(w)] = e_me_{m+1}e_{m-1} - e_{m+1}e_{m-1}e_m + e_{m-1}e_{m+1}e_m - e_me_{m-1}e_{m+1} =\\
&e_m[e_{m+1},e_{m-1}] - [e_{m+1},e_{m-1}]e_m = [e_m,[e_{m+1},e_{m-1}]]
\end{aligned}$$

Analogously we have

$$\operatorname*{Res}_{zyw}\left[(zyw)^{-m}(z+w)(y^2-zw)f(z)f(y)f(w)\right] = [f_m,[f_{m+1},f_{m-1}]].$$

So last two relations state that

$$[e_m,[e_{m+1},e_{m-1}]] = 0,$$
$$[f_m,[f_{m+1},f_{m-1}]] = 0.$$

5. The isomorphism between $\mathcal{E}$ and $\tilde{\mathcal{E}}$.

5.1. **The map $\phi: \tilde{\mathcal{E}} \to \mathcal{E}$.** In this subsection we construct a surjective map $\phi: \tilde{\mathcal{E}} \to \mathcal{E}$. We know that the algebra $\tilde{\mathcal{E}}$ is generated by elements e_k, f_k and $h_n^{\pm}$. Let us put $\phi(e_k) = u_{(1,k)}$, $\phi(f_k) = u_{(-1,k)}$ and $\phi(h_n^{\pm}) = \theta_{(0,\pm n)}$. We need to check that ϕ respects sets of relations for $\tilde{\mathcal{E}}$ and $\mathcal{E}$, i.e. elements u_x for $x \in \{(1,k),(-1,k),(0,n)\}$ satisfy relations of $\tilde{\mathcal{E}}$.

Let us consider formal series

$$\mathbb{T}_1(z) = \sum_{l\in\mathbb{Z}} u_{1,l} z^l,$$
$$\mathbb{T}_{-1}(z) = \sum_{l\in\mathbb{Z}} u_{-1,l} z^l,$$
$$\mathbb{T}_0^1(z) = 1 + \sum_{n\in\mathbb{Z}_{>0}} \theta_{0,n} z^n,$$
$$\mathbb{T}_0^{-1}(z) = 1 + \sum_{n\in\mathbb{Z}_{>0}} \theta_{0,-n} z^{-n},$$
$$\delta(z) = \sum_{l\in\mathbb{Z}} z^l.$$

To show that a map of algebras ϕ is well-defined it is enough to check the following set of relations

(i) $\mathbb{T}_0^{\epsilon_1}(z)\mathbb{T}_0^{\epsilon_2}(w) = \mathbb{T}_0^{\epsilon_2}(w)\mathbb{T}_0^{\epsilon_1}(z)$

(ii) $\chi(z,w)\mathbb{T}_0^{\epsilon_2}(z)\mathbb{T}_{\epsilon_1}(w) = -\chi(w,z)\mathbb{T}_{\epsilon_1}(w)\mathbb{T}_0^{\epsilon_2}(z)$,

(iii) $\chi(z,w)\mathbb{T}_\epsilon(z)\mathbb{T}_\epsilon(w) = -\chi(w,z)\mathbb{T}_\epsilon(w)\mathbb{T}_\epsilon(z)$,

(iv) $[\mathbb{T}_{-1}(z), \mathbb{T}_1(w)] = \dfrac{1}{\alpha_1}\delta(\dfrac{z}{w})(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z))$,

(v) $[u_{(1,m)}, [u_{(1,m+1)}, u_{(1,m-1)}]] = 0$,

(vi) $[u_{(-1,m)}, [u_{(-1,m+1)}, u_{(-1,m-1)}]] = 0$.

Proposition 5.1.1. *The map $\phi : \tilde{\mathcal{E}} \to \mathcal{E}$ is a well-defined surjective map of algebras.*

Proof. From the relations of the EHA $\mathcal{E}$ we have $u_{(0,l)}u_{(0,m)} = u_{(0,m)}u_{(0,l)}$ and therefore $\mathcal{E}^0$ is a commutative subalgebra. By the definition $\mathbb{T}_0^\epsilon \in \mathcal{E}^0$. The relation (i) follows.
We will prove (ii) for $\epsilon_1 = \epsilon_2 = \{+\}$. All other cases are analogous. We need to rewrite this condition. We put $\zeta(x) = \frac{(1-q_1x)(1-q_2x)}{(1-x)(1-qx)}$.

Lemma 5.1.2. $\zeta(x) = \exp\left(\sum_n \frac{x^n(1-q_1^n)(1-q_2^n)}{n}\right)$.

Proof. Let us consider $\log(\zeta(x))$. Note that $\log(1-z) = -\sum_n \frac{z^n}{n}$.

$$\log(\zeta(x)) = \log(1-q_1x) + \log(1-q_2x) - \log(1-x) - \log(1-q_2x) =$$
$$-\sum_n q_1^n\frac{x^n}{n} - \sum_n q_2^n\frac{x^n}{n} + \sum_n \frac{x^n}{n} + \sum_n q^n\frac{x^n}{n} = \sum_n \frac{(1-q_1^n)(1-q_2^n)}{n}x^n.$$

Taking exponent of both sides we get the proposition. □

Note that

$$\frac{\chi(z,w)}{\chi(w,z)} = \frac{(z-q_1w)(z-q_2w)(z-q^{-1}w)}{(w-q_1z)(w-q_2z)(w-q^{-1}z)} = \frac{z^2(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})(z-q^{-1}w)}{w^2(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})(w-q^{-1}z)} =$$
$$\frac{z^2}{w^2}\frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{q^{-1}z(q\frac{w}{z}-1)}\frac{q^{-1}w(q\frac{z}{w}-1)}{(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})} = \frac{w}{z}\frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{(1-q\frac{w}{z})(z-w)}\frac{(1-q\frac{z}{w})(z-w)}{(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})} =$$
$$\frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{(1-q\frac{w}{z})(1-\frac{w}{z})}\frac{(1-q\frac{z}{w})(\frac{z}{w}-1)}{(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})} = -\frac{\zeta(w\backslash z)}{\zeta(z\backslash w)}.$$

The relation (ii) can be rewritten as

$$\mathbb{T}_0^+(z)\mathbb{T}_1(w)\zeta(\frac{z}{w}) = \mathbb{T}_1(w)\mathbb{T}_0^+(z)\zeta(\frac{w}{z}).$$

This relation should be understood as an equality of coefficients for these formal series expanded in $|w| > |z|$. Let us look at

$$\zeta(\frac{w}{z}) = \frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{(1-\frac{w}{z})(1-q\frac{w}{z})} = \frac{(\frac{z}{w}-q_1)(\frac{z}{w}-q_2)}{(\frac{z}{w}-1)(\frac{z}{w}-q)} = \frac{(q_1^{-1}\frac{z}{w}-1)(q_2^{-1}\frac{z}{w}-1)}{(\frac{z}{w}-1)(q^{-1}\frac{z}{w}-1)} = \frac{(1-q_1^{-1}\frac{z}{w})(1-q_2^{-1}\frac{z}{w})}{(1-\frac{z}{w})(1-q^{-1}\frac{z}{w})} =$$
$$\exp\left(\sum_n \frac{(1-q_1^{-n})(1-q_2^{-n})}{n}\left(\frac{z}{w}\right)^n\right)$$

Then we have the relation (ii) in the following form

$$\mathbb{T}_0^+(z)\mathbb{T}_1(w)\exp\left(\sum_n \frac{z^n(1-q_1^n)(1-q_2^n)}{nw^n}\right) = \mathbb{T}_1(w)\mathbb{T}_0^+(z)\exp\left(\sum_n \frac{z^n(1-q_1^{-n})(1-q_2^{-n})}{nw^n}\right),$$
$$\mathbb{T}_1(w)\mathbb{T}_0^+(z) = \mathbb{T}_0^+(z)\mathbb{T}_1(w)\exp\left(\sum_n \left(\frac{(1-q_1^n)(1-q_2^n)}{n} - \frac{(1-q_1^{-n})(1-q_2^{-n})}{n}\right)\left(\frac{z}{w}\right)^n\right).$$

Note that $\mathbb{T}_0^+(z) = \exp\left(\sum_n \alpha_n u_{(0,n)} z^n\right)$.

$$\mathbb{T}_1(w)\exp\left(\sum_n \alpha_n u_{(0,n)} z^n\right) =$$
$$\exp\left(\sum_n \alpha_n u_{(0,n)} z^n\right)\mathbb{T}_1(w)\exp\left(\sum_n \left(\frac{(1-q_1^n)(1-q_2^n)}{n} - \frac{(1-q_1^{-n})(1-q_2^{-n})}{n}\right)\left(\frac{z}{w}\right)^n\right).$$

Lemma 5.1.3. *Suppose that $[e,p] = ec \Rightarrow e\exp(p) = \exp(p)e\exp(c)$.*

Proof. We need to show that $e\frac{p^k}{k!} = \sum_{a+b=k}\frac{p^a}{a!}e\frac{c^b}{b!}$. We will prove it by the induction by k.
For $k=1$ we have $e = pe + ec$.
Suppose that the statement holds for $k-1$. Then

$$e\frac{p^k}{k!} = \frac{1}{k}ep^{k-1}p = \frac{1}{k}\sum_{a+b=k-1}\frac{p^a}{a!}e\frac{c^b}{b!}p = \frac{1}{k}\sum_{a+b=k-1}\frac{p^a}{a!}ep\frac{c^b}{b!} =$$
$$\frac{1}{k}\sum_{a+b=k-1}\frac{p^{a+1}}{a!}e\frac{c^b}{b!} + \frac{1}{k}\sum_{a+b=k-1}\frac{p^a}{a!}e\frac{c^{b+1}}{b!} = \frac{1}{k}\sum_{a+b=k}\left(\frac{p^a}{(a-1)!}e\frac{c^b}{b!} + \frac{p^a}{a!}e\frac{c^b}{(b-1)!}\right) =$$
$$\frac{1}{k}\sum_{a+b=k}(a+b)\frac{p^a}{a!}e\frac{c^b}{b!} = \sum_{a+b=k}\frac{p^a}{a!}e\frac{c^b}{b!}.$$

□

Let us compute $[\mathbb{T}_1(w), \sum_n \alpha_n u_{(0,n)} z^n]$.

$$[\mathbb{T}_1(w), \sum_n \alpha_n u_{(0,n)} z^n] = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_{>0}} \alpha_n [u_{(1,k-n)}, u_{(0,n)}] w^{k-n} z^n =$$
$$\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_{>0}} \alpha_n u_{(1,k)} w^k \left(\frac{z}{w}\right)^n = \mathbb{T}_1(w)\sum_{n\in\mathbb{Z}_{>0}}\alpha_n\left(\frac{z}{w}\right)^n = \mathbb{T}_1(w)\sum_{n\in\mathbb{Z}_{>0}}\frac{(1-q_1^n)(1-q_2^n)(1-q^{-n})}{n}\left(\frac{z}{w}\right)^n =$$
$$\mathbb{T}_1(w)\sum_{n\in\mathbb{Z}_{>0}}\frac{(1-q_1^n)(1-q_2^n)-(1-q_1^{-n})(1-q_2^{-n})}{n}\left(\frac{z}{w}\right)^n.$$

Applying Lemma 5.1.3 we have the proof of (ii).
The only known proof of (iii) uses the isomorphism $\mathcal{E}$ with the EHA of elliptic curve and GL_n modular forms. The first fact will be covered in future talks. The second one will lead us too far away from our topic. One can read details in Kapranov's paper [K].
To prove (iv) let us consider the coefficient of $z^k w^l$ in expressions of both sides.

$$\underset{z^k w^l}{C}\left([\mathbb{T}_{-1}(z), \mathbb{T}_1(w)]\right) = [u_{(-1,k)}, u_{(1,l)}],$$
$$\underset{z^k w^l}{C}\left(\frac{1}{\alpha_1}\delta(\frac{z}{w})(c^{-1}\mathbb{T}_0^-(z) - c\mathbb{T}_0^+(z))\right) = \frac{1}{\alpha_1}\underset{z^{k+l}}{C}\left(c^{-1}\mathbb{T}_0^-(z) - c\mathbb{T}_0^+(z)\right).$$

We need to consider three cases.

1) $k + l < 0$,

$$[u_{(-1,k)}, u_{(1,l)}] = \kappa_{\alpha((-1,k),(1,l)}\frac{\theta_{(0,l+k)}}{\alpha_1} ,$$
$$\alpha((-1,k),(1,l) = (\frac{(1,-k)+(1,l)-(0,-k-l)}{2}) = (1,l) \Rightarrow [u_{(-1,k)}, u_{(1,l)}] = c\frac{\theta_{(0,l+k)}}{\alpha_1},$$
$$\frac{1}{\alpha_1}\underset{z^{k+l}}{C}\left(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)\right) = \frac{1}{\alpha_1}\underset{z^{k+l}}{C}(c\mathbb{T}_0^-(z)) = c\frac{\theta_{(0,l+k)}}{\alpha_1}.$$

2) $k + l = 0$,

$$[u_{(-1,k)}, u_{(1,l)}] = [u_{(-1,-l)}, u_{(1,l)}] = \frac{c - c^{-1}}{\alpha_1},$$
$$\frac{1}{\alpha_1}\underset{z^{k+l}}{C}(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)) = \frac{c - c^{-1}}{\alpha_1}.$$

3) $k + l > 0$,

$$[u_{(-1,k)}, u_{(1,l)}] = -\kappa_{\alpha((-1,k),(1,l)}\frac{\theta_{(0,l+k)}}{\alpha_1},$$
$$\alpha((-1,k),(1,l) = -(\frac{(1,-k)+(1,l)-(0,k+l)}{2}) = (-1,k) \Rightarrow [u_{(-1,k)}, u_{(1,l)}] = -c^{-1}\frac{\theta_{(0,l+k)}}{\alpha_1},$$
$$\frac{1}{\alpha_1}\underset{z^{k+l}}{C}\left(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)\right) = -\frac{1}{\alpha_1}\underset{z^{k+l}}{C}(c^{-1}\mathbb{T}_0^+(z)) = -c^{-1}\frac{\theta_{(0,l+k)}}{\alpha_1}.$$

We see that in all cases

$$\underset{z^k w^l}{C}\left([\mathbb{T}_{-1}(z), \mathbb{T}_1(w)]\right) = \underset{z^k w^l}{C}\left(\frac{1}{\alpha_1}\delta(\frac{z}{w})(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z))\right) \text{ for all } k, l \Rightarrow$$
$$[\mathbb{T}_{-1}(z), \mathbb{T}_1(w)] = \frac{1}{\alpha_1}\delta(\frac{z}{w})(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)).$$

To prove (v) we note that the triangle $(0, (1, l+1), (2, 2l))$ has no interior lattice points. Then $[[u_{(1,l+1)}, u_{(1,l-1)}], u_{(1,l)}] = [u_{(2,2l)}, u_{(1,l)}] = 0$.
The proof of (vi) is analogous.
Lemma 3.2.1 implies surjectivity of ϕ. □

We are ready to state the main theorem of this talk.

Theorem 5.1.4. *The map ϕ gives an isomorphism between $\tilde{\mathcal{E}}$ and the EHA $\mathcal{E}$.*

5.2. **Properties of $\tilde{\mathcal{E}}$.** The main goal of this subsection is to get the similar result to Lemma 3.5.1 for the algebra $\tilde{\mathcal{E}}$. Let us introduce subalgebras $\tilde{\mathcal{E}}^>$ generated by elements e_l, $\tilde{\mathcal{E}}^0$ generated by elements $h_{\pm n}$ and $\tilde{\mathcal{E}}^<$ generated by elements f_l. We have the following triangular decomposition.

Proposition 5.2.1. *The multiplication map $\tilde{m} : \tilde{\mathcal{E}}^> \otimes \tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^< \to \tilde{\mathcal{E}}$ is surjective.*

Proof. It is enough to establish that

$$1)[e_k, f_l] \in \tilde{\mathcal{E}}^0,$$
$$2)[e_k, h_{\pm n}] \in \tilde{\mathcal{E}}^> + m'(\tilde{\mathcal{E}}^> \otimes \tilde{\mathcal{E}}^0), \text{ where } m' : \tilde{\mathcal{E}}^> \otimes \tilde{\mathcal{E}}^0 \to \tilde{\mathcal{E}} \text{ is the multipliction map.}$$
$$3)[f_k, h_{\pm n}] \in \tilde{\mathcal{E}}^< + m''(\tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^<), \text{ where } m'' : \tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^< \to \tilde{\mathcal{E}} \text{ is the multipliction map..}$$

Then the lemma will follow from the same reason as in Lemma 3.5.1. We will prove these facts using the relations of $\tilde{\mathcal{E}}$.
The relation (iv) implies that $[e(z), f(w)] \in \tilde{\mathcal{E}}^0(z, w)$ Note that coefficient of $z^k w^l$ is exactly $[e_k, f_l]$ that proves 1).
Let us prove 2) for h_n. The h_{-n} case is analogous.

Lemma 5.2.2. *We have $[e_l, h_n] = \beta e_{l+n} + \sum_i e_i h_i$ for some e_i, h_i. Analogously $[e_l, h_{-n}] = \beta' e_{l-n} + \sum_i e_i h_i$.*

Proof. First, let us compute the coefficients of $z^k w^l$ in the relation (ii) of $\tilde{\mathcal{E}}$.

$$h_{k-3}e_l - (q_1 + q_2 + q^{-1})h_{k-2}e_{l-1} + (q_1^{-1} + q_2^{-1} + q)h_{k-1}e_{l-2} - h_k e_{l-3} =$$
$$e_l e_{k-3} - (q_1^{-1} + q_2^{-1} + q)e_{l-1}e_{k-2} + (q_1 + q_2 + q^{-1})e_{l-2}e_{k-1} - e_{l-3}e_k.$$

We can rewrite it as

$$[h_{k-3}, e_l] + (q_1^{-1} + q_2^{-2} + q - q_1 - q_2 - q^{-1})e_{l-1}h_{k-2} + (q_1^{-1} + q_2^{-1} + q)[e_{l-1}, h_{k-2}]-$$
$$(q_1^{-1} + q_2^{-2} + q - q_1 - q_2 - q^{-1})e_{l-2}h_{k-1} - (q_1^{-1} + q_2^{-1} + q)[e_{l-2}, h_{k-1}] - [h_k, e_{l-3}] = 0,$$
$$q_1^{-1} + q_2^{-2} + q - q_1 - q_2 - q^{-1} = (1 - q_1)(1 - q_2)(1 - q^{-1}) = \alpha_1, \text{ so}$$
$$[h_{k-3}, e_l] + \alpha_1 e_{l-1}h_{k-2} + (q_1^{-1} + q_2^{-1} + q)[e_{l-1}, h_{k-2}]-$$
$$\alpha_1 e_{l-2}h_{k-1} - (q_1^{-1} + q_2^{-1} + q)[e_{l-2}, h_{k-1}] - [h_k, e_{l-3}] = 0,$$

Note that in the formula above we suppose $h_{-1} = h_{-2} = \ldots = 0$ and $h_0 = 1$ because we compute commutator with $\psi^+(z)$.
Now we are ready to prove this proposition. We use induction by n. For $n = 1$ we have from the relation above

$$[e_{l-3}, h_1] = \alpha_1 e_{l-2}h_0 = \alpha_1 e_{l-2}.$$

Suppose thaw we prove the proposition for all $n < l$. From the computation above $[e_k, h_l]$ is a linear combination of $[e_{k+1}, h_{l-1}]$, $[e_{k+2}, h_{l-2}]$, $[e_{k+3}, h_{l-3}]$, $[e_{k+1}h_{l-1}]$ and $[e_{k+2}h_{l-2}]$. The proposition follows. □

3) is analogous to 2). □

Now we have the following commutative diagram

$$\begin{tikzcd}
\tilde{\mathcal{E}}^> \otimes \tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^< \arrow[r, "\hat{\phi}"] \arrow[d, "\tilde{m}"] & \mathcal{E}^> \otimes \mathcal{E}^0 \otimes \mathcal{E}^< \arrow[d, "m"] \\
\tilde{\mathcal{E}} \arrow[r, "\phi"] & \mathcal{E}
\end{tikzcd}$$

From Lemma 3.5.1 the map m is isomorphism. Proposition 5.2.1 states that $\tilde{m}$ is surjective. Proposition 5.1.1 shows that ϕ is surjective. Therefore Theorem 5.1.4 is equivalent to the fact that $\hat{\phi}$ is an isomorphism.

Remark 5.2.3. *Theorem 5.1.4 implies that $\tilde{m}$ is an isomorphism.*

We will denote $\hat{\phi}$ also by ϕ. It is enough to check that every part $\phi^>$, ϕ^0 and $\phi^<$ is an isomorphism. For commutative subalgebras $\mathcal{E}^0$ and $\tilde{\mathcal{E}}^0$ the isomorphism is obvious. Cases of $^>$ and $^<$ are analogous, so we will focus on the first of them. We need to introduce more combinatorical notions.

5.3. **Minimal paths.** We are going to introduce the notion of a minimal path.
For a point $z \in Z^\times$ we have a line L going through $(0,0)$ and z. We want to choose a closest parallel to L line $L' \neq L$ that has lattice points. We have two identical options. We set L' be a line lying above L. By a *minimal path* we denote the path $(x, z-x)$ with $x \in L'$. We need two important lemmas about minimal paths.

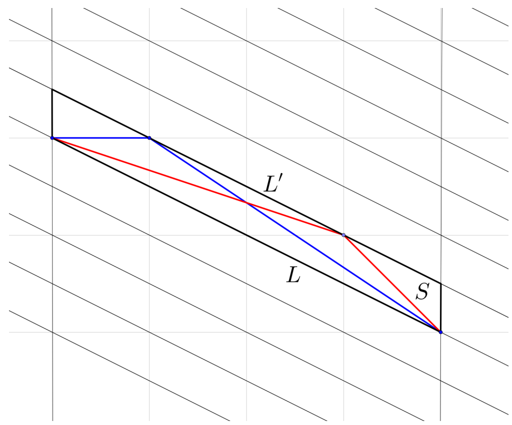

Figure 7. Two different minimal paths.

Lemma 5.3.1. *Let $z = (r,d)$ be a positive segment with $r \geq 2$. Then there exists a minimal path $(x, z-x)$ such that* $\mathrm{rank}(x) > 0$ *and* $\mathrm{rank}(z-x) > 0$. *In fact there are* $\gcd(r,d)$ *such paths.*

Proof. Let S be a strip bounded by the vertical lines through origin and z and lines L and L'. Then to prove the lemma it is enough to find a lattice point on L' inside S.
If $\deg(z) > 1$ the statement is obvious. So it is enough to consider $\deg(z) = 1$. The only case when we don't have a point on the line L' inside S is when both intersection points of vertical lines through $(0,0)$ and z with L' are lattice points. We know that $r \geq 2$ and $\deg(z) = 1$, so we have a point $(1, \frac{d}{r}) \in L$ such that $\frac{d}{r} \notin \mathbb{Z}$. Then consider a line L'' parallel to L through $(1, \lceil \frac{d}{r} \rceil)$. It is closer to L then L', so $(x, z-x)$ was not a minimal path and we get a contradiction. □

Lemma 5.3.2. *A positive path $(x, z-x)$ is minimal if and only if* $\deg(x) = \deg(z-x) = 1$ *and the triangle $\Delta_{x,z-x}$ has no interior lattice points.*

Proof. All points inside the triangle $\Delta_{x,z-x}$ or on the sides x and $z-x$ are closer to the line L then L' containing x. Therefore if the path $p = (x, z-x)$ is minimal then there are no such lattice points.
Suppose that we have a not-minimal path $p = (x, z-x)$ satisfying conditions of lemma. Let us choose a minimal path $p = (y, z-y)$. Note that $S(\Delta_{x,z-x}) > S(\Delta_{y,z-y})$ because they have common side z but a point y is closer to L then x. On the other hand by Pick's formula $S(\Delta_{x,z-x}) = \deg(z) = S(\Delta_{y,z-y})$. We get a contradiction. □

Note that for a positive minimal path $(x, z-x)$ we have $[u_x, u_{z-x}] = \frac{\theta_z}{\alpha_1}$.

5.4. **The proof of the main theorem.** In this subsection we will prove Theorem 5.1.4. We have a map $\phi : \tilde{\mathcal{E}}^> \to \mathcal{E}^>$. Note that for every r we have a vector space $\mathcal{E}_r^>$ generated by the paths u_p with $\mathrm{rank}(p) = r$. We are going to construct for each $p \in \mathcal{E}_r^>$ a preimage $e_p = \phi^{-1}(u_p)$ satisfying commutation relations of $\mathcal{E}$ and show that such e_p (for all r) generate $\tilde{\mathcal{E}}$ as vector space. We prove it by the induction on r.

For $r=1$ we have $e_{1,m}=e_m=\phi^{-1}(u_{(1,m)})$ by the definition of ϕ.
Now suppose that we define such preimages for all paths of rank k for all $k<r$. Let us put $e_{k,m}=\phi_{r-1}^{-1}(u_{(k,m)})$. Our goal is to define a preimage of $u_{(r,m)}$. We set $z=(r,m)$ and choose a minimal path $(x,z-x)$. Then we have $[u_x,u_{z-x}]=u_z$. We want to set $e_z=[e_x,e_{z-x}]$. We need the following lemma.

Lemma 5.4.1. *For any two minimal paths $p=(x,z-x)$ and $p'=(y,z-y)$ it holds $[e_x,e_{z-x}]=[e_y,e_{z-y}]$.*

Proof. Set $l=\deg(z)$ and $z_0=\frac{z}{l}=(r,d)$. As in the definition of minimal path let L be the line through the origin and z and L' – the closest parallel to L line above it with lattice points on it. We can choose the point x to be closest to the vertical line through $(0,0)$ lattice point on L'. If $r>1$ Then all minimal paths are (x,lz_0-x), $(x+z_0,(l-1)z_0-x)$, …, $(x+(l-1)z_0,z_0-x)$. Note that if $l=1$ there is a unique minimal path, so the lemma holds. So we may suppose $l>1$. We have already defined element e_{z-z_0}, so $[e_{x+(i-1)z_0},e_{(l-i)z_0-x}]=e_{z-z_0}=[e_{x+iz_0},e_{(l-i-1)z_0-x}]$ holds. We apply the operator $ad(e_{z_0})$ to both sides of the equation.

$$\begin{aligned}
[e_{z_0},[e_{x+(i-1)z_0},e_{(l-i)z_0-x}]]&=[[e_{z_0},e_{x+(i-1)z_0}],e_{(l-i)z_0-x}]-[[e_{z_0},e_{(l-i)z_0-x}],e_{x+(i-1)z_0}]=\\
&-[e_{x+iz_0},e_{(l-i)z_0-x}]+[e_{x+(i-1)z_0},e_{(l-i+1)z_0-x}],\\
[e_{z_0},[e_{x+iz_0},e_{(l-i-1)z_0-x}]]&=[[e_{z_0},e_{x+iz_0}],e_{(l-i-1)z_0-x}]-[[e_{z_0},e_{(l-i-1)z_0-x}],e_{x+iz_0}]=\\
&-[e_{x+(i+1)z_0},e_{(l-i-1)z_0-x}]+[e_{x+iz_0},e_{(l-i)z_0-x}],\\
[e_{x+(i-1)z_0},e_{(l-i+1)z_0-x}]-[e_{x+iz_0},e_{(l-i)z_0-x}]&=[e_{x+iz_0},e_{(l-i)z_0-x}]-[e_{x+(i+1)z_0},e_{(l-i-1)z_0-x}].
\end{aligned}$$

We need just one additional relation to show that $[e_x,e_{lz_0-x}]=[e_{x+iz_0},e_{(l-i)z_0-x}]$ for any i. Let us consider three cases.
1) $l=2$. We have $[e_x,e_{z_0-x}]=e_{z_0}$. Applying $ad(e_{z_0})$ we get

$$0=[e_{z_0},[e_x,e_{z_0-x}]]=[[e_{z_0},e_x],e_{z_0-x}]+[[e_{z_0},e_{z_0}-x],e_x]=-[e_{x+z_0},e_{z_0-x}]+[e_x,e_{2z_0}-x].$$

2) $l\geq 3$. By the induction hypothesis $[e_x,e_{(l-2)z_0-x}]=[e_{x+iz_0},e_{(l-i-2)z_0-x}]=e_{(l-2)z_0}$. We apply the operator $ad(e_2z_0)$. Note that by Lemma 5.3.2 the triangle $0,x+iz_0,(l-i)z_0-x$ has no interior lattice points and no lattice points on sides $x+iz_0$ and $(l-i)z_0-x$. Therefore a triangle $\Delta_{x+iz_0,2z_0}$ has no interior lattice points (it lies inside a parallelogram on sides z and $x+iz_0$) and no lattice points on the side $x+iz_0$.

$$\begin{aligned}
[e_{2z_0},[e_x,e_{(l-2)z_0-x}]]=&[[e_{2z_0},e_x],e_{(l-2)z_0-x}]-[[e_{2z_0},e_{(l-2)z_0-x}],e_x]=\\
&-[e_{x+2z_0},e_{(l-2)z_0-x}]+[e_x,e_{lz_0-x}],\\
[e_{2z_0},[e_{x+iz_0},e_{(l-i-2)z_0-x}]]=&[[e_{2z_0},e_{x+iz_0}],e_{(l-i-2)z_0-x}]-[[e_{2z_0},e_{(l-i-2)z_0-x}],e_{x+iz_0}]=\\
&-[e_{x+(i+2)z_0},e_{(l-i-2)z_0-x}]+[e_{x+iz_0},e_{(l-i)z_0-x}],\\
[e_x,e_{lz_0-x}]-[e_{x+2z_0},e_{(l-2)z_0-x}]=&[e_{x+iz_0},e_{(l-i)z_0-x}]-[e_{x+(i+2)z_0},e_{(l-i-2)z_0-x}].
\end{aligned}$$

From this relation and all obtained by applying $ad_{e_{z_0}}$ we have $[e_x,e_{lz_0-x}]=[e_{x+iz_0},e_{(l-i)z_0-x}]$ for all i, q.e.d. □

For any sequence $s=(x_1,x_2,\ldots,x_n)$ we set $e_s=e_{x_1}e_{x_2}\ldots e_{x_n}$. We need to check that by this definition we get the same element for two equivalent (i.e. representing the same path) sequences s and s'.

Lemma 5.4.2. *If s and s' are two equivalent sequences then $e_s=e_{s'}$.*

Proof. It is enough to consider the case of $l(s)=2$. Indeed suppose that we have proved for the case of two segments and let $l(s)>2$. For every two segments x_i, x_{i+1} in $s=(x_1,\ldots,x_n)$ we have $e_s=e_{s'}$ with $s'=(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n)$. Let us have $s=(kw,lw)$ and $s'=(lw,kw)$. Let us take two minimal paths $p=(x,(k+l)w-x)$ and $p'=(x+lw,kw-x)$. Note that from Lemma

5.4.1 we have $[e_x, e_{(k+l)w-x}] = [e_{x+lw}, e_{kw-x}]$ and from the induction hypothesis $[e_x, e_{kw-x}] = \frac{\theta_{kw}}{\alpha_1}$ with $\theta(z)$ defined from e_x in the same way as for the EHA $\mathcal{E}$.

$$\begin{aligned}[\theta_{kw}, e_{lw}] = \alpha_1([[e_x, e_{kw-x}], e_{lw}]) =& \alpha_1([e_x, [e_{kw-x}, e_{lw}]] + [[e_x, e_{lw}], e_{kw-x}]) = \\ & \alpha_1(-[e_x, e_{(k+l)w-x}] + [e_{x+lw}, e_{kw}]) = 0.\end{aligned}$$

But $\theta_{kw} = \alpha_k e_{kw} + t$ where t is a linear combination of e_p with $l(p) \geq 2$. As u_p for convex p are linearly independent we have $[e_{kw}, e_{lw}] = 0$. □

In such way for every convex path p of rank r we construct an element e_p such that $\phi(e_p) = u_p$. We put J be a vector subspace of $\tilde{\mathcal{E}}^{>}$ generated by all e_p. From Proposition 3.4.6 u_p is a basis of $\mathcal{E}^{>}$, so ϕ gives an isomorphism between J and $\mathcal{E}^{>}_r$. It remains to show that $e_p \in J$ for every (not necessarily convex) path p of rank r. We will use the induction by area $a(p)$. If $a(p) = 0$ then p is convex path and the statement holds. Suppose that we have already shown it for all p with $a(p) < n$.

It is enough to consider the case of $l(p) = 2$. If $l(p) > 2$ then for every subpath p' of two segments $a(p') < a(p)$, so we can choose a convexification with smaller area by the induction hypothesis. The same argument on the area function shows that after finite number of convexifications we will get a linear combination of convex paths.

For a path $p = (x, (r,d)-x)$ we consider the region R bounded by the line L going through the origin and (r,d), parallel line through x and vertical lines though $(0,0)$ and (r,d). The path p divides this region in 3 segments Δ, Δ_x and Δ'. Let us consider two different cases.
1) There is a lattice point y in one of two triangles Δ and Δ'. We allow this point lie on the left boundary of Δ or right boundary of Δ' (green on Figure 8) but not on the boundary in common with Δ nor on the top boundary (black on Figure 8). Assume that $y \in \Delta$ and consider paths $p' = (y, (r,d)-y)$ and $m = (y, x-y, (r,d)-x)$. By construction we have $a(q) < a(p)$.

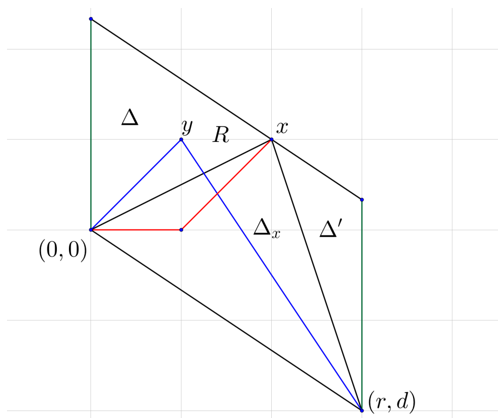

FIGURE 8. Three triangles Δ, Δ_x, Δ', the path p' (blue) and the convexification q (red).

Proposition 5.4.3. *$e_m = \beta e_p + t = \beta' e_{p'} + t'$ where $t, t' \in J$ and $\beta, \beta' \in \mathbb{C}(q_1, q_2, c)$.*

Proof. Proposition 3.4.2 we have $u_y u_{x-y} = \beta u_x + \sum_q \beta_q u_q$ where q runs among the set of convexifications of $(y, x-y)$. Note that $\mathrm{rank}(y) < r$ and $\mathrm{rank}(x-y) < r$, so by the induction hypothesis the same holds for $e_y e_{x-y}$. We have

$$e_m = e_y e_{x-y} e_{(r,d)-x} = \beta e_x e_{(r,d)-x} + \sum_q \beta_q e_q e_{(r,d)-x} = \beta e_p + \sum_q \beta_q e_{q'},$$

where q' is the concatenation of q and $(r,d)-x$. Note that $l(q') \geq 3$ and $a(q') \leq a(p)$ because q' is a local convexification of p. Therefore by the induction hypothesis $e_{q'} \in J$. The first equality of the proposition holds.
The equality $e_m = \beta' e_{p'} + t'$ is proved in a similar way.

$$e_m = e_y e_{x-y} e_{(r,d)-x} = \beta' e_y e_{(r,d)-y} + \sum_q \beta_q e_y e_q = \beta e_{p'} + \sum_q \beta_q e_{q'},$$

where $q' \in J$ by the same reasons. □

Therefore we may suppose that p has no interior lattice points in Δ and Δ'.
2) If both Δ and Δ' have no interior lattice points then the same is true for Δ_x that is equal to the sum of these two triangles reflected along common sides. Suppose that Δ_x has lattice points on boundaries,
(i) We have a point on the bottom boundary. Then we have a point z on the upper boundary of the region R. We put $q = (z, (r,d)-z)$ and $q^\sharp = ((r,d)-z, z)$. The computation above shows that $e_p = \beta e_q + t$ where $t \in J$. By the construction triangle $\Delta_{z,(r,d)-z}$ has no interior lattice points and $\deg(z) = \deg((r,d)-z) = 1$. Therefore $u_q = u_{q^\sharp} + \frac{\theta_{(r,d)}}{\alpha_1} \in J$. So we may assume that $\deg((r,d)) = 1$.
(ii) In subsection 3.2.4 we show that if there are no interior lattice points in $\Delta_{x,y}$ then either $\deg(x) = 1$ or $\deg(y) = 1$ or $\deg(x) = \deg(y) = 2$. We apply this statement to the triangle $\Delta_x = \Delta_{x,(r,d)-x}$. In the last case we have $\deg((r,d)) \geq 2$. So we may suppose that $\deg(x) = 1$.
Set $w = \frac{(r,d)-x}{\deg((r,d)-x)}$ and put $y = x - w$. Suppose that y is positive. The triangle $\Delta_{y,w}$ has no interior lattice points because its area is the same as the area of $\Delta_{x,w}$. The area $S(\Delta_{y,(r,d)-x}) =$

$S(\Delta_{x,(r,d)-x})$, so $\Delta_{y,(r,d)-x}$ has no interior lattice points. By the induction hypothesis we get $[e_y, e_w] = e_x,\ [e_y, e_{(r,d)-x}] = e_{(r,d)-w}$. Therefore

$$[e_x, e_{(r,d)-x}] = [[e_y, e_w], e_{(r,d)-x}] = [[e_y, e_{(r,d)-x}], e_w] + [e_y, [e_w, e_{(r,d)-x}]] = [e_{(r,d)-w}, e_w] = \frac{\theta_{(r,d)}}{\alpha_1} \in J,$$

because $((r,d) - w, w)$ is a minimal path. It finishes the proof of the theorem 5.1.4.

Remark 5.4.4. *In fact $y = x - w$ can be not a positive segment. But the cases of $\mathcal{E}^>$ and $\mathcal{E}^<$ are analogous and* $\operatorname{rank}(y) < r$*, so we may apply the induction hypothesis to both subalgebras. Suppose that* $\operatorname{rank}(y) = l$*. We get:*

$[e_y, e_w] = c^{-l} e_x,\ [e_y, e_{(r,d)-x}] = c^{-l} e_{(r,d)-w}$. *Therefore*

$$[e_x, e_{(r,d)-x}] = c^{-l}[[e_y, e_w], e_{(r,d)-x}] = c^{-l}[[e_y, e_{(r,d)-x}], e_w] + c^{-l}[e_y, [e_w, e_{(r,d)-x}]] =$$
$$c^{-l}[e_{(r,d)-w}, e_w] = \frac{\theta_{(r,d)}}{\alpha_1} \in J.$$

6. Hopf algebra structure.

In terms of Drinfeld generators $\mathbb{T}_1$, $\mathbb{T}_0^{\pm}$ and $\mathbb{T}_{-1}$ it is easy to write down the bialgebra structure on $\mathcal{E}$.

$$\Delta(\mathbb{T}_1(z)) = \mathbb{T}_1(z) \otimes 1 + \mathbb{T}_0^+(z) \otimes \mathbb{T}_1(z),$$
$$\Delta(\mathbb{T}_{-1}(z)) = \mathbb{T}_{-1}(z) \otimes 1 + \mathbb{T}_{-1}(z) \otimes \mathbb{T}_0^-(z),$$
$$\Delta(\mathbb{T}_0^{\pm}(z)) = \mathbb{T}_0^{\pm}(z) \otimes \mathbb{T}_0^{\pm}(z).$$

It will be shown in next talks that the algebra $\mathcal{E}$ admits a Hopf algebra structure.

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